Optimal employment in frictional business cycles and intertemporal discontinuity of production and internal prices

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The present paper studies the optimal behavior of the firm facing labor friction characterized by convex hiring cost and sufficiently differentiated output good. The problem turns out to be a singular control problem. General paths are studied including socially inefficient ones caused by the rise of demand constraint. Analysis of optimal firing behavior becomes possible with this setup. It turns out that the singularity brings costate jumps at junction points to the firing phase, which result in the discontinuity of production. Those jumps have been known to occur in problems with inequality constraints that contain only state variables. The present model shows that they can occur with inequality constraints with control variables too when singularity is present.

I. Introduction

The present paper studies the optimal behavior of the firm facing labor friction characterized by convex hiring cost and sufficiently differentiated output good. Even though there are conflicting arguments on empirical applicability of convex hiring cost, from the point of view that hiring cost is a variation of adjustment cost, it should be natural to assume it from the theoretical perspective.[1] The convexity in hiring cost makes the labor adjustment process time-consuming, which in turn determines the level of income and demand for output at every moment while the adjustment process is still going on. When output goods are sufficiently differentiated, for such an adjustment process to be fulfilled, expectation on the social choice of the equilibrium path must be shared uninterruptedly among firms that the economy ultimately reaches to the long-term equilibrium. If expectation is not well-coordinated, the investment in labor by a single firm according to the efficient path will cause a loss, since any discount in output prices will not retrieve the cost of investment. It raises multiple inefficient paths that depend on

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[1] Yashiv (2000) and Blatter, Muehlemann and Schenker (2012) support convexity. On the other hand, Abowd and Kramarz (2003) and Kramarz and Michaud (2010) finds concavity for long-term workers in France. However, the hiring cost in the latter is defined to be directly observable ones which differs from our setup that the cost arises from the wage payment to workers internally allocated to the hiring sector and arises from decreasing returns in both production and hiring sectors.

the degree of coordination. Yashiv (2006, 2007) analyzed the efficient equilibrium in an economy with convex hiring cost. The present paper extends them to those inefficient cases. In the sense that expectation is the main driving force of business cycles, the model shares the spirit with news-driven business cycle models by Beaudry and Portier (2004) and others. However, in our model, the source of expectation is not limited to technological news on productivity fluctuation, but it is more generalized shared view about future equilibrium characterized by nature of beauty contest.

A firm operating in an economy with labor friction has an intertemporal horizon. However, a model of a going concern with no firing cost easily falls in the category of singular control problems. They are the cases in which Hamiltonian becomes independent from some control variables and the (ordinal procedure) of the maximum principle does not allow to find the optimal control value.[2] It occurs when the switching function becomes zero. Even though there were cases that singularity is artificially evaded in early times as Johnson and Gibson (1963) pointed out, obviously it is not guaranteed that those singularity should be "pathological". The model studied in this paper shows that singularity is indeed not only unexcludable but comprises important part of business cycles. Namely, the firing phase is singular. Moreover, since the singular control is derived from the resulting state constraint on the boundary, the optimal control in the firing phase is determined only in a derivative form, leaving determination of the initial condition out of the firing phase. It brings intertemporal discontinuity in costate variables and thus in the output level. This is a similar property to optimal control problems with state variable inequality constraints (SVICs), which is basically brought by the truncatability of the problem into subperiods. It is known that the occurrence time of costate discontinuity in state-constrained problems is generally indeterminate between entering and leaving times.[3] It will be proved that our setup shows the discontinuity in both times.

The properties of the model with no firing cost gives a good implication for a model with firing cost. It tells us that firing is optimal to visit in a chunk on the first day of the firing phase in general. With firing cost, such a discontinuous behavior is too costly. The firm is better to keep idle employment as far as the cost to hold it measured in its absolute value (which is negative) is less than the firing cost. Thus, it brings labor hoarding.

Section II explains the model. Section III analyzes steady states both with unbounded and bounded demand constraints. Section IV and V study out of steady states. Section IV examines firing phase and Section V examines the entering and exiting from those phases. Section VI examines implication to labor hoarding. Section VII studies how labor hoarding behavior changes when there is convex firing cost. Section VIII shows that linear firing cost mixes results of the

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[2]Rozonoer (1959).

[3]This is true for many of problems. Hartl, Sethi and Vickson (1995) points out a case in which the discontinuity occurs within the boundary intervals.

basic model and that of Section VII. Section IX examines the implication on wage bargaining and the endogenization of bargaining process in the firm. Section X concludes.

II.   The model with no firing regulation: a singular control problem

Consider an economy with differentiated output goods produced by labor. Goods are perishable. The labor market is frictional. Firms engage in two activities, production of goods and hiring of future workers. The production function of goods is denoted by $f(\hat{l})$ where $\hat{l} \geq 0$ is the measure of employment in the production sector. The firm faces hiring friction, that is hiring is not an autonomous unattended process but requires expense of internal resources to get hiring results. The hiring activity of the firm is expressed by an independent production function $g(\tilde{l}; \theta)$ where $\tilde{l} \geq 0$ is the measure of employment in the hiring sector and θ is a parameter which represents vu ratio in the labor market. In harmless places, we just omit θ to represent it $g(\tilde{l})$ and treat it as a time-varing function. It can be shown that the optimal firm efficiently utilizes existent employment in the no firing cost case. So for the moment we simply assume that the sum of the employment in the production and hiring sectors equates the total employment,[4] i.e. $\hat{l} + \tilde{l} = l$ where l is the total employment of the firm. Both f and g obey decreasing returns, Inada conditions and $f(0) = g(0; \theta) = 0$. It implicitly assumes the presence of hidden fixed inputs such as real estate. Suppose the total cost of hidden input factors is c. This is a simplification assumption to focus on the employment behavior of the firm. The decreasing returns of $g(\cdot)$ represents convex hiring cost. The employees separate from the firm at instantaneous rate $\sigma_t > 0$. In addition, the firm can intentionally reduce the employment by firing denoted by $x_t \geq 0$. Therefore, the transition of labor becomes

$$(1) \qquad \dot{l}_t = g(\tilde{l}_t; \theta_t) - \sigma_t\, l_t - x_t.$$

The real wage rate is denoted by a right-continuous function of time $w_t > 0$. It is assumed to be bargained between the industry representative and the industry-wide labor union so that the process of wage determination stays exogenous to the firm. This assumption makes the analysis considerably simple. The endogenization of the wage bargaining process is yielded to another paper. The firm discounts future real profits by instantaneous rate $r > 0$. Since hiring is the investment decision made by the firm, it must have intertemporal horizon. Then, denoting the value as of time t when initial labor is l by $\Pi(t, l)$, the value of the representative firm as of present time $t = 0$ becomes

$$\Pi(0, l_0) = \max_{\hat{l}, x} \int_0^\infty \left(f(\hat{l}) - w\, l - c \right) \mathrm{e}^{-rt} \mathrm{d}t$$

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[4]This assumption will be removed as we introduce the firing cost later.

which is subject to the labor transition (1) and the demand constraint

$$(2) \qquad\qquad\qquad f(\hat{l}) \leq y$$

and the following constraints on control variables.

$$(3) \qquad\qquad\qquad 0 \leq \hat{l} \leq l$$
$$(4) \qquad\qquad\qquad 0 \leq x < \infty$$

Two points need to be mentioned. First, the demand constraint (2) is assumed to be potentially binding, in which $y > 0$ is the demand directed to the firm which would have been met if there were not the hiring cost, and it is assumed to be continuously time-differentiable. The presence of this inequality is a reflection of insufficiency in price mechanism in the time-consuming adjustment process of sufficiently differentiated output good. It should be emphasized that immediate adjustment of labor is suboptimal with convex hiring cost, so the *adjustment process involves time where actual production is undertaken at any moment*. In general, y is a function of lifetime income, the rate of interest and, in off-equilibrium context, output price, given the capacity limit imposed by the hiring technology. The level of lifetime income depends on the degree of expectation coordination among firms, since attempts to break the constraint by a single firm will fail. It requires coordinated move by firms of positive measure to increase output breaking the binding constraint.[5] Conditions that the demand constraint becomes effective will be explained later in this section. Since coordinated output accompanies the distribution of realized income earned by firms, y can be interpreted as effective demand in the sense that the demand is not just planned but endorsed by purchasing power. With the current assumption of perishable output good, y is simply current income in equilibrium that reflects the degree of coordination among firms.

Second, even though the firing variable x enters linearly in Hamiltonian, finite upper bound technically required by a regular linear problem will be proved unnecessary in (4) *except for the initial adjustment*. If initial employment is extremely large for some reason, there will be massive firing at $t = 0$ resulting in discrete adjustment of employment, which implies $x_0 = +\infty$. It can be heuristically confirmed by placing limit of a sequence of problems with upper bound $\bar{x} < +\infty$ in admissible x and taking $\bar{x} \uparrow +\infty$. It imposes an assumption that catastrophe never happens in the limit. However, in a rigorous treatment, it does not conform to the framework of the maximum principle. If such a treatment is necessary, we can start the problem after initial discrete adjustment is already done if it were to exist. Namely, initial employment l_0 is assumed to be in the domain that the optimal employment path starting from it is guaranteed to be

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[5]If such a move is not unanimous and if there is any theoretical structure to make it an equilibrium, it brings heterogeneity in output prices and market shares.

continuous. This setup makes the problem solvable by the maximum principle. Anyway except for the initial discrete adjustment which can potentially happen, the optimal firing turns out to be singular, i.e. x is not a bang-bang.[6] The proof of the following claim will be represented after optimal conditions are characterized in Section IV.

CLAIM 1: *Suppose y is continuously differentiable except for the junction times from/to the firing phase and x faces a constraint $x \in [0, \bar{x}]$ where $\bar{x} < \infty$. If $\bar{x}$ is set sufficiently large corresponding to the maximum variation of y, the optimal x is never bound by $\bar{x}$ except for the initial adjustment.*

Costate variable of l is denoted by λ, Lagrangean of demand constraint by $\mu \geq 0$ and that of $\hat{l} \leq l$ by $\eta \geq 0$. The optimal dynamics of the costate variable becomes

$$(5) \qquad \dot{\lambda} = -\left[g'(l - \hat{l}) - (r + \sigma) \right] \lambda + w - \eta.$$

for any time at which l is differentiable. Since y is assumed to be strictly positive and f satisfies Inada conditions, $\hat{l} > 0$ always holds. Then, the first order condition becomes

$$(6) \qquad (1 - \mu) f'(\hat{l}) = \lambda g'(l - \hat{l}) + \eta$$

and for firing, by imposing arbitrarily large $\bar{x} > 0$,

$$(7) \qquad x = \begin{cases} 0 & \text{if } \lambda > 0 \\ [0, \bar{x}] & \text{if } \lambda = 0 \\ \bar{x} & \text{if } \lambda < 0. \end{cases}$$

Together with (5) and (6), the optimal condition when $\hat{l} \leq l$ is not binding can be expressed as

$$(8) \qquad (r + \sigma) \lambda = (1 - \mu) f'(\hat{l}) - w + \dot{\lambda}.$$

This form of optimal costate dynamics states that, when the value of the firm is viewed as abstract capital, its marginal cost equates the sum of instantaneous effective marginal profits and marginal capital gain. To see it, integrate equation (8) through l. Then, the left-hand side of (8) becomes $(r + \sigma) \Pi$ and its right-hand side becomes $\int (1 - \mu) f'(\hat{l}) \mathrm{d}l - wl + \dot{\Pi}$. Namely, the left-hand side of the integrated form is the corporate cost of capital with separation premium σ. Its right-hand side is the instantaneous profits plus capital gain where the revenue is discounted by the shadow price of demand constraint. (8) claims that these are equated marginally in optimum.

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[6]For a singular control problem, see Johnson and Gibson (1963) and Kelly, Kopp and Moyer (1967).

To see the condition that breaking the demand constraint (2) is unprofitable and thus it persists in the problem as a constraint, focus on the time immediately after the entering time to the constraint and denote the individual demand curve at that time by $z = d(p, r, \psi)$ where z is quantity of output good *in circulation in the market*, p is the individual output price, r is market interest rate and ψ is the current "average" income per firm given future income. Normalizing total measure of firms to be one, ψ can be regarded as the aggregate income with simplification assumption that demand schedule to each differentiated good is equal. Then $d(1, r, \psi) = \psi$ holds. The output good in circulation or the *ex post* supply of product can be expressed as $z = \min\{f(\hat{l}), d(p, r, \psi)\}$. If $f(\hat{l}) > d(p, r, \psi)$, the difference $f(\hat{l}) - d(p, r, \psi)$ is actually produced but does not circulate in the market. Let the demand constraint be binding at y. To violate the demand constraint at the entering time, the firm needs to reduce its price by $dp < 0$ whereas producing $dy = f'(\hat{l})\,dl$ with increased employment dl. The firm receives the increase of demand by $(\partial d/\partial p)\,dp + [(\partial d/\partial \psi)/n]\,dy$ assuming n equally differentiated firms exist in the economy, i.e. each firm receives $1/n$ of total demand, and thus $d\psi = dy/n$. Increase of production cost accompanying this action is $wdl > wf^{-1\prime}(y)\,dy$ since $d\hat{l}/dl < 1$. Taking increase of output *in circulation* z as denominator and letting $n \to +\infty$, marginal revenue is $p + d\,(\partial d/\partial p)^{-1}$ for all domain. On the other hand, marginal cost for $z > y$ is $w\,dl/dz = w\,dl/d\psi > nwf^{-1\prime}(w)$ which diverges in this domain, and that for $z \leq y$ is $w\,dl/dz = w/(f'(\hat{l})\,d\hat{l}/dl)$. Note that wage payment to the hiring sector is included in the cost. Figure 1 depicts the argument above in an analogous fashion to Chamberlin (1962). It is important to emphasize that
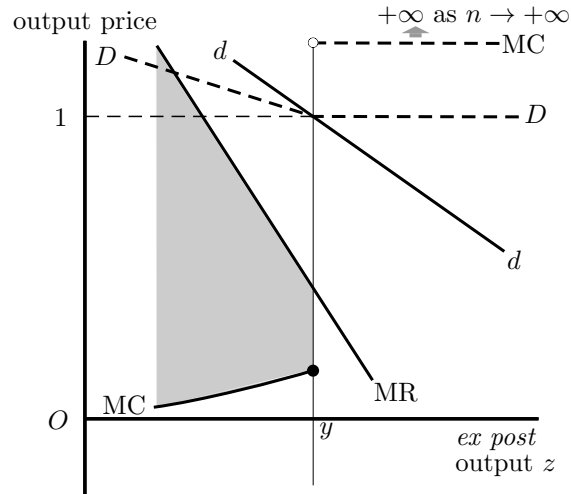

Figure 1. : Reluctance of transcending the effective demand

the adjustment process is time consuming so that income level is determined in

each moment of adjustment. In off-equilibrium context, the firm regains control of output price since the bargaining of real wage rate is based on *market* output price and individual output price is allowed to differ in off-equilibrium,[7] so that the individual output price appears in the vertical axis as a control variable. The main departure from the familiar diagram of monopolistic competition is that the horizontal axis takes output *in circulation* z, i.e. output met by demand. Since the object of measurement is the cost for off-equilibrium actions, the diagram should be understood as drawing *ex post* equilibrium z for each action. Compared to the case in which horizontal axis is taken to be planned output, one unit in the latter is transformed to $1/n$ units in the latter only in $z > y$ domain. The complementary domain is equivalently transformed. By taking horizontal axis in this way, the demand curve drawn here can be understood as the *effective demand schedule*. Namely, successful expansion of industrial supply accompanies increase of income and demand, but individual expansion without followers will not. The former extends the domain $z \leq y$ but the latter do not. The industry demand curve is drawn as a kinked curve DD. It is a smooth curve on the planned axis, but as a result of the above transformation, it is drawn as a kinked curve in the diagram. dd shows the individual demand curve. It is a smooth curve on this diagram but should be kinked on the planned axis. The cost structure is asymmetric between expansion and reduction of supply as derived above. Reflecting strictly positive profits in equilibrium, the marginal cost curve stays below the marginal revenue curve in $z \leq y$ domain, but it jumps above the demand curve if n is large enough in domain $z > y$. Integrating the optimal condition (8) in off-equilibrium context gives

$$\lambda = \int_0^\infty (\mathrm{MR} - \mathrm{MC})\, f'(\hat{l})\, \mathrm{e}^{-(r+\sigma)t} \mathrm{d}t$$

where the right hand side goes negative as n becomes large. It implies additional employment reduces the value of firm. Therefore, no individual firms have incentive to transcend the shared belief on the demand upper bound. The phrase *sufficiently differentiated output good* we have already used without definition means that n is large enough so that the marginal cost in $z > y$ domain becomes strictly greater than the marginal revenue.

It is worthwhile examining what happens in the case of non-differentiated output goods, i.e. the case of $n = 1$. In such a case, reduced output price will attract infinite amount of demand which grants the attempt of constraint violation. All firms will do so, which implies that contemporaneous real wages settled by bargaining are unchanged, demand constraint is broken, and ex-post interest rate rises only by ignorable degree. The above argument suggests that the existence of demand constraint is peculiar to industrialized goods, not homogeneous ones

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[7]Analogous to the argument made by Arrow (1959) that competitive firms behave as monopolists in off-equilibrium, even though real wage rate is fixed by bargaining in equilibrium in this model, firms have control on price setting in off-equilibrium state.

such as agricultural goods.

In the present paper, we want to focus on time-discontinuity of production level and that of μ which can be interpreted as the impact of unexpected variation of y on the value of the firm, in addition to showing optimal behavior of the firm facing convex hiring cost. So we start with the following proposition.

PROPOSITION 1: *If $\lambda > 0$, then μ is time-continuous and λ is continuously differentiable on the optimal path.*

PROOF:

If demand constraint is not binding in any open domain in time, then $\mu \equiv 0$ on that domain so time-continuity of μ on the domain is obvious. Suppose demand constraint is binding in any open domain in time. Perturbation specification of the problem à la Bryson, Denham and Dreyfus (1963) would be

$$
d\Pi(t_i, l_{t_i}) = \int_{t_i}^{t_{i+1}} \left(\frac{\partial H}{\partial l} \delta l + \frac{\partial H}{\partial \hat{l}} \delta \hat{l} + \frac{\partial H}{\partial x} \delta x \right) e^{-r(t-t_i)} dt
$$
$$
- \int_{t_i}^{t_{i+1}} \lambda \, e^{-r(t-t_i)} \, \delta \dot{l} \, dt + \left(\frac{\partial \Pi(T, l_{t_{i+1}})}{\partial l_{t_{i+1}}} - \lambda_{t_{i+1}} \right) e^{-r(t_{i+1}-t_i)} \delta l_{t_{i+1}}
$$

which should be stationary at optimum for i-th subinterval after dividing $[0, \infty)$ into finite subintervals where H is the current value Hamiltonian, $i = 0, 1, 2, \ldots, N$, $t_0 = 0$ and $t_{N+1} = \infty$. To minimize the subdivision of time, let N be the number of indifferentiability of l in the time scale so that l is indifferentiable on t_i. This subdivision of time scale is necessary because the second term of the right hand side of the above equation needs to be integrable by parts to obtain (5). It requires the time derivative of l to exist. Namely, (5) is applicable only to the interior time of each subinterval. Now, choose $t \in \{s : \lambda_s > 0\}$. Then, $x = 0$ and $\hat{l}$ is differentiable from $\hat{l} = f^{-1}(y)$ and differentiability of y, which implies l is differentiable from (1). It implies t is interior time of subintervals and indeterminate function λ_t can be safely defined as a differentiable function via integration by parts of the second term of the above equation so that (5) holds. Then, the first-order condition (6) implies time-differentiability of μ on the domain. Finally, suppose that the demand constraint is binding at time T under consideration and there exists a point where the constraint is unbinding in any open neighborhood of T with radius $\varepsilon > 0$, which is denoted by $N_\varepsilon(T)$. Then, corresponding to a sequence of ε converging to zero, we can take converging subsequence of time $t \in N_\varepsilon(T)$ at any of which $f'(\hat{l}_t) = \lambda \, g'(l_t - \hat{l}_t)$ holds from the first-order condition (6) when $\lambda > 0$ and also $f(\hat{l}_t) \leq y_t$ holds. On the other hand, if $\mu_T > 0$, then (6) implies $\hat{l}_T < \lim_{t \to T} \hat{l}_t$ from the continuity of l and λ. Since $y_T = f(\hat{l}_T) < \lim_{t \to T} f(\hat{l}_t) \leq \lim_{t \to T} y_t$, it contradicts to the continuity of y. Therefore $\mu_T = 0$. It implies that μ is continuously connected at time T when transiting from domain $\mu > 0$ to $\mu = 0$ or vice versa. Finally, (5) and (5) obtain

$\dot{\lambda} = (r + \sigma)\lambda - [(1 - \mu)f'(\hat{l}) - w]$. If μ is continuous, it implies λ is continuously differentiable. $\square$

III. Steady states

Steady states occur with $\lambda > 0$, obviously with no firing. Since the present model has external input y, there exist multiple steady states depending on it. As explained in the previous section, individual firms do not have incentive to break the demand constraint when outputs are sufficiently differentiated so y is a representation of the degree of coordination among firms. It is shown that in steady states with demand constraint in effect, relaxation of the constraint implies the increase of the effective marginal value of employment.[8]

1. Unconstrained steady state

This is the efficient steady state characterized by $\mu = 0$. $\dot{\lambda} = 0$ in the costate dynamics (5) gives

$$\lambda_{\mathrm{ss}} = \frac{w}{g'(l_{\mathrm{ss}} - \hat{l}_{\mathrm{ss}}) - (r + \sigma)} > 0,$$

which implies $x_{\mathrm{ss}} = 0$ where subscript "ss" represents the value at the unconstrained steady state. Inada conditions guarantee that the first-order condition (6) provides an interior solution when $\lambda > 0$ so we have $f'(\hat{l}_{\mathrm{ss}}) = \lambda_{\mathrm{ss}} g'(l_{\mathrm{ss}} - \hat{l}_{\mathrm{ss}})$. Together with $\dot{l} = 0$ in labor transition (1), $(l_{\mathrm{ss}}, \hat{l}_{\mathrm{ss}})$ is characterized to solve

$$(9) \qquad\qquad f'(\hat{l}_{\mathrm{ss}}) = \frac{g'(l_{\mathrm{ss}} - \hat{l}_{\mathrm{ss}})}{g'(l_{\mathrm{ss}} - \hat{l}_{\mathrm{ss}}) - (r + \sigma)} w$$

$$(10) \qquad\qquad g(l_{\mathrm{ss}} - \hat{l}_{\mathrm{ss}}) = \sigma\, l_{\mathrm{ss}},$$

the solution of which exists when $r > 0$.[9] Note that (9) implies $f'(\hat{l}_{\mathrm{ss}}) > w$. The marginal productivity of labor in the production sector is strictly greater than the marginal cost of labor simply because some labor is absorbed in the hiring activity. If hiring efficiency rises, then the difference between f' and w shrinks.

2. Constrained steady states

Constrained steady states are inefficient steady states characterized by binding demand constraint with $\mu > 0$. They satisfy $f(\hat{l}_{\mathrm{css}}) = y$ where subscript "css"

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[8]As a benchmark, in spot production with demand constraint, the effective marginal value becomes the present value of wages. It is derived from the fact that, in each moment, $\max_l f(l) - w\,l$ subject to $f(l) \leq y$ implies $w = (1 - \mu)\,f'(l)$ with $\mu \geq 0$.

[9]If $r \leq g'(L) - \sigma < 0$ where L is the solution in $g(L) = \sigma L$ where all employment is dedicated to hiring, the solution may not exist. However, since negative discount rate is unlikely to continue forever, we can safely eliminate such a case. It implies $g'(L) > r + \sigma$. By decreasing returns of g, $g'(l - \hat{l}) > r + \sigma$ holds for any unconstrained/constrained steady states.

represents the value at constrained steady states. For their existence, y must be a constant function. Since stationarity requires the employment in the hiring sector to be strictly positive, we have $\eta = 0$. Also with stationarity in labor transition, the constrained steady state for given y is characterized by followings:

$$(11) \qquad \lambda_{\mathrm{css}} = \frac{w}{g'\big(l_{\mathrm{css}} - f^{-1}(y)\big) - (r + \sigma)} > 0$$

$$(12) \qquad g\big(l_{\mathrm{css}} - f^{-1}(y)\big) = \sigma\, l_{\mathrm{css}}$$

The relation between unconstrained and constrained steady states is depicted in Figure 2. The curve labeled $\dot{\lambda} = 0$ shows the costate steady state condition for
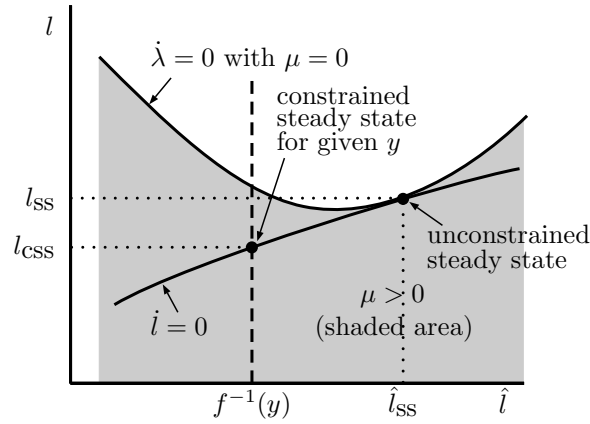

Figure 2. : Unconstrained and constrained steady states

unconstrained case (9). The curve labeled $\dot{l} = 0$ shows the labor steady state conditions (10) and (12). Since $f^{-1}(y) \leq \hat{l}_{\mathrm{css}}$, stationarity conditions for labor (10) and (12) imply that $l_{\mathrm{css}} \leq l_{\mathrm{ss}}$ the inequality of which is strict except for the obvious case $f(\hat{l}_{\mathrm{css}}) = y$. Therefore, the curve $\dot{l} = 0$ always locates below the curve $\dot{\lambda} = 0$ for any $\hat{l} \leq \hat{l}_{\mathrm{ss}}$. Unconstrained steady states locate on the curve $\dot{l} = 0$.

λ_{css} given by (11) is expressed in another form $\lambda_{\mathrm{css}} = [(1-\mu)f'(\hat{l}_{\mathrm{css}}) - w]/(r+\sigma)$. It reads the marginal value of employment for the firm is the discounted present value of $(1 - \mu)f'(\hat{l}_{\mathrm{css}}) - w$ in which the discount rate is affected by separation premium. In this place, $(1 - \mu)f'(\hat{l})$ is termed the *effective marginal productivity* of the production sector. Facing the demand constraint, effective price of output is discounted by the shadow price of the constraint. Hiring sector has costate variable λ as its internal output price, which is also indirectly affected by present and future shadow price of the demand constraint.

PROPOSITION 2:   *At a constrained steady state, increase in demand y brings 1)*

increase in total employment and in both production and hiring sectors, 2) increase in marginal value of employment, 3) increase in effective marginal productivity of the production sector and increase in marginal productivity of the hiring sector, and 4) decrease in demand duals.

PROOF:

1) Applying the implicit function theorem to (12) obtains $dl_{css}/dy = g'/[f'(g' - \sigma)] > 0$. On the other hand, $d\hat{l}_{css}/dy = 1/f' > 0$. They result in

$$(13) \qquad \frac{d\tilde{l}_{css}}{dy} = \frac{dl_{css}}{dy} - \frac{d\hat{l}_{css}}{dy} = \frac{\sigma}{g' - \sigma}\frac{1}{f'} > 0.$$

2) Similar application of the theorem to (11) derives $d\lambda_{css}/d\tilde{l}_{css} = -g''\lambda_{css}^2/w$, from which (13) implies

$$\frac{d\lambda_{css}}{dy} = -\frac{g''\lambda_{css}^2}{w}\frac{\sigma}{g' - \sigma}\frac{1}{f'} > 0.$$

3) From the above result,

$$\begin{aligned}
\frac{d\lambda_{css}g'(\tilde{l}_{css})}{dy} &= \left(\lambda_{css}g'' + \frac{d\lambda_{css}}{d\tilde{l}_{css}}g'\right)\frac{d\tilde{l}_{css}}{dy} \\
&= \left(1 - \frac{g'\lambda_{css}}{w}\right)g''\lambda\frac{d\tilde{l}_{css}}{dy} \\
&= (r + \sigma)\frac{d\lambda_{css}}{dy} > 0
\end{aligned}$$

where $\lambda_{css}g'(\tilde{l}_{css})$ is the marginal productivity of the hiring sector and from the first-order condition (6), it equates to the effective marginal productivity $(1 - \mu)f'(\hat{l}_{css})$.

4) Since $1 - \mu_{css} = \lambda_{css}g'(\tilde{l}_{css})/f'(\hat{l}_{css})$,

$$\frac{d(1 - \mu_{css})}{dy} = \frac{r + \sigma}{f'}\frac{d\lambda_{css}}{dy} - \frac{\lambda_{css}g'f''}{(f')^3} > 0,$$

which brings $d\mu_{css}/dy < 0$. □

The result that the effective marginal productivity and the marginal value of labor increase may worth attention. Depending on the wage bargaining process, they can imply the rise of wages. It means that increase in both wages and employment coexist without technological progress. The last result of the proposition shows that the impact of the unexpected variation of demand on the firm's value becomes smaller as the demand level becomes higher.

IV. Firing

Now, we embark on untidy non-steady states. Since the transition of labor is linear in firing, we need precise treatment on its edge. Employment becomes redundant if both current and future production requires smaller labor, which reflects in y and λ. Under the assumption that y is differentiable, the adjustment of labor is fulfilled within continuous variation, except for the case that initial employment is out of feasible range with any optimal paths. With feasible initial employment, firing is done with $\lambda = 0$ which implies singularity of the problem, i.e. a non-standard procedure is required to obtain the optimal path.

In the following description, the following notations are used for subsets on the time scale. Define the firing phase X_t the maximum connected set on time scale including t where $x_s^* > 0$ for any $s \in X_t$ and the asterisk shows the optimal path. Formally, $X_t := \bigcup_i \{I_i \ni t : x_s^* > 0, \forall s \in I_i\}$ where I_i is a closed interval. If $x_t^* = 0$, then $X_t = \emptyset$. The closure of X_t writes in an interval form $\bar{X}_t = [t^E, t^L]$ where t^E is the entering time in X_t and t^L is the leaving time from X_t. Although they are defined interval-wise, they are used without mentioning it if the objective interval is obvious. Similarly, the non-hiring phase Λ_t^o is defined to be the maximum connected set on time scale including t where $\lambda_s^* = 0$ for any $s \in \Lambda_t^o$, i.e. $\Lambda_t^o := \bigcup_i \{I_i \ni t : \lambda_s^* = 0, \forall s \in I_i\}$. If $\lambda_t^* > 0$, then $\Lambda_t^o = \emptyset$. In the interval form, $\bar{\Lambda}_t^o = [t^e, t^l]$ where the entering and leaving time from/to Λ_t^o are denoted by t^e and t^l, respectively. Again, the same abuse of notation applies as $\bar{X}_t$. Also, define $\Lambda_t^- := \bigcup_i \{I_i \ni t : \lambda_s^* < 0, \forall s \in I_i\}$ and $W := \{t : l_t^* < f'^{-1}(w_t)\}$. Union of those intervals are denoted without subscripts by $X := \bigcup_t X_t$, $\Lambda^o := \bigcup_t \Lambda_t^o$ and $\Lambda^- := \bigcup_t \Lambda_t^-$. Denote the set of all t^E's, t^L's, t^e's and t^l's by E^X, L^X, E^Λ and L^Λ, respectively.

PROPOSITION 3: $X_t \subset \Lambda_t^o$ for any $t \in W$.

PROOF:

The result is obvious if $X_t = \emptyset$. Suppose $X_t \neq \emptyset$. Define $\bar{x} = \sup_t(-\dot{y}_t/f'(l_t))$. Suppose that the demand constraint holds with equality, i.e. $f(\hat{l}_t) = y_t$ for some $t \in \Lambda_t^- \cap W$ such that $\Lambda_t^- \neq \emptyset$. Optimal $x_t = \bar{x}$ leads to $\mu_s = 0$ for any $s \in \Lambda_t^-$ such that $s > t$ since $f'(l_t)\dot{l}_t < \dot{y}_t$. Since μ_s is right-continuous, $\mu_t = 0$ and $\dot{\lambda}_s$ exists for such s and $t + 0$.[10] From (5) and (6), costate transition becomes $\dot{\lambda}_s = (r+\sigma)\lambda_s - f'(l_s) + w_s < 0$ for any $s \geq t$. The relation is recursively justified starting from $s = t$ so that $s \in \Lambda_t^-$ implies $\dot{l}_s < 0$ which means $s + \varepsilon \in W$ and thus $s + \varepsilon \in \Lambda_t^- \cap W$ for arbitrarily small $\varepsilon \geq 0$. It ultimately causes $\lim_{s \to \infty} l_s < 0$. Therefore, any path which enters $\Lambda_t^- \cap W$ cannot be optimal. It implies that $X_t \cap W \nsubseteq \Lambda_t^- \cap W$ whereas $X_t \subseteq \Lambda_t^o \cup \Lambda_t^-$ for any $X_t \neq \emptyset$. Therefore, $X_t \cap W \subseteq \Lambda_t^o \cap W$. □

The above proposition proves Claim 1 and firing x becomes a singular control, not a bang-bang:

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[10]See Theorems 4.1 and 4.2 of Hartl, Sethi and Vickson (1995).

COROLLARY 1: *Firing x_t is a singular control if $t \in W$.*

PROOF:

Since l is right-differentiable, $|X_t| > 0$ if $X_t \neq \emptyset$. From Proposition 3, it implies $|\Lambda_t^o| > 0$. Namely, if $x_t > 0$, then $\lambda_t = 0$ and $\lambda_{t+\varepsilon} = 0$ for arbitrarily small $\varepsilon > 0$. Since λ_t is a switching function of x_t, which is zero on an interval with positive measure whenever $x_t > 0$, x_t is a singular control. □

The proposition suggests that, for a massive firing, i.e. discontinuous decrease of employment, to happen, it should occur only once at the very beginning of the economy, which implies that we can safely separate such a phase from the analysis and concentrate on the dynamics after the negative "big-bang". Since $\bar{x}$ is an artificial boundary, if such a transition were to happen, it finishes instantaneously by a discontinuous decrease of employment, which can be understood as the limit of dynamics when $\bar{x} \to \infty$. Even if continuously large non-autonomous force acts in the middle, it will not bring the system to the initial big-bang state since sufficiently large continuous change of employment absorbs such a shock. Figure 3 shows a phase diagram for an unbounded autonomous case. The manifold
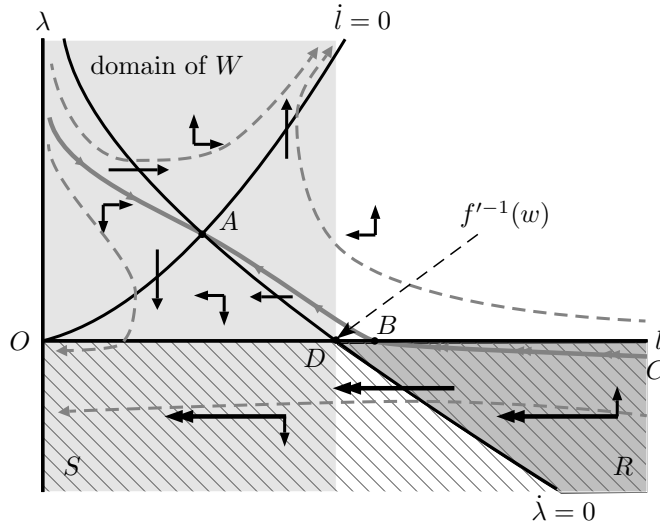

Figure 3. : Phase diagram for an unbounded autonomous case

$\dot{l} = 0$ is an upward sloping curve stable in terms of l that passes through the neighborhood of the origin. That of $\dot{\lambda} = 0$ is downward sloping, unstable in terms of λ, so that $\lambda \to +\infty$ holds as $l \to 0$ on it and passes through $l = f'^{-1}(w)$. In the forth quadrant shown as a hatched area, there is "strong" leftward flows. The flow instantaneously reaches to the goal as $\bar{x} \to \infty$. There is a saddle path to the unique non-zero steady state A in this particular unbounded autonomous

case. The saddle path in the forth quadrant is drawn as almost flat downward sloping curve in the figure. However, since $\bar{x}$ is arbitrary, it should be understood that it converges to the horizontal axis as $\bar{x} \to \infty$, the optimal firing on the path is $x^* \to +\infty$. The same holds for any paths which reaches to any points on the horizontal axis except interval OD. In non-autonomous case, the optimal path is generally different from the one drawn in the diagram because phase diagram itself transforms. If the optimal path in such a case were in the hatched light grey area S in Figure 3, the only way that it survives as optimal is to move to the hatched dark grey area R. In other cases, the path trespasses on the negative employment region due to $\dot{\lambda} < 0$. Move from S to R is generally impossible in the autonomous case as the vector field in the figure shows. However, non-autonomous cases require a check whether sufficiently quick fluctuation of the boundary of domain of W caused by external forces does not actually allow such a move. If allowed, the path may come back to the non-hatched area surviving as optimal. The following proposition formalizes that it never happens even in non-autonomous cases.

PROPOSITION 4 (Impossibility of negative big-bang in the middle): *If $t \in W$, then $\lambda_s \geq 0$ for any $s > t$ along the optimal path.*

PROOF:
   Suppose $\lambda_v = 0$ at $v > t$ and $\lambda_{v+\varepsilon} < 0$ for arbitrarily small $\varepsilon > 0$. Then, from the transition (5) of λ, right-continuity of μ and continuity of w and l, $(1-\mu_{v+\varepsilon})f'(l_{v+\varepsilon}) > w_{v+\varepsilon}$ must hold for $\lambda_{v+\varepsilon} < 0$, which implies $v+\varepsilon \in W$. Then, since x is arbitrarily large in (1), $\dot{l}$ is always smaller than any non-autonomous change of $f'^{-1}(w)$, which implies the optimal path never enters W^c from $W \cap \Lambda^-_{v+\varepsilon}$, thus diverging to $l_s \to -\infty$ as $s \to \infty$. Thus, it is excluded from the optimal path. $\square$

The proposition can be restated as follows with the optimal control (7).

COROLLARY 2: *$x_t > 0$ occurs only if $\lambda_t = 0$ for any $t \in W$ along the optimal path.*

Based on the singularity of firing, the next proposition shows that it occurs only when the demand constraint is binding and $\dot{y} < 0$.

PROPOSITION 5 (Optimal controls and effectiveness of the demand constraint): $\hat{l}^*_t = l^*_t$ and $\tilde{l}^*_t = 0$ for $t \in \Lambda^o \cap W$.

$$(14) \qquad\qquad\qquad x_t = -\sigma l_t - \frac{\dot{y}_t}{f'(l_t)}$$

and $\mu_t > 0$ for any $t \in (\mathring{\Lambda}^o \cup E^{\Lambda}) \cap W$. Moreover, μ_t is differentiable for any $t \in \mathring{\Lambda}^o \cap W$ and right-differentiable for $t \in E^{\Lambda}$. Also

$$(15) \qquad\qquad\qquad\qquad f(l^*_t) = y^*_t$$

for any $t \in \bar{\Lambda}^o \cap W$ and $\dot{y}_t < 0$ for any $t \in \Lambda^o \cap W$.

PROOF:

$\lambda_t = \dot{\lambda}_t = 0$ for any $t \in \Lambda^o \cap W$ where $\dot{\lambda}$ is right-derivative at $t = t^e$ and left-derivative at $t = t^l$. Then, costate dynamics (5) implies $\eta = w > 0$ and also $\hat{l} = l$ and $\tilde{l} = 0$ by complementarity. The first order condition (6) obtains

$$(16) \qquad (1 - \mu) f'(l) = \eta = w.$$

It implies $\mu_t > 0$ for any $t \in \mathring{\Lambda}^o \cap W$ since $f'(l_t) > w_t$ for $t \in W$. (16) holds for $t^e \in W$ from the right-continuity of μ and the continuity of l and w, so $\mu_{t^e} > 0$. Together with continuity of l, it implies that $f(l_t^*) = y_t$ holds for any $t \in \bar{\Lambda}^o \cap W$. Since $\dot{l}_t = -\sigma l_t - x_t = \dot{y}_t / f'(l_t)$ holds for any $t \in \Lambda^o \cap W$, optimal firing is given by (14). Since $x_t \geq 0$, it implies $\dot{y}_t \leq -\sigma l_t f'(l_t) < 0$. It also implies differentiability of μ_t for any $t \in \mathring{\Lambda}^o \cap W$ and right-differentiability for $t \in E^\Lambda$ by the differentiability of l and w and $f \in C^1$. $\qquad \square$

Suppose that $B_t := \Lambda_t^o \setminus X_t \neq \emptyset$ for some t and $s \in B_t$. If B_t consists of an interval, then Proposition 5 implies $\dot{l} = -\sigma l$ and thus $\dot{y} = -\sigma l f'(l)$ need to hold on that interval. However, it is generically true that actual y will not satisfy this condition so that we can safely assume $\bar{X}_t = \Lambda_t^o$ almost surely if both are not empty and if y is considered to be randomly chosen by nature.

V. Entering and leaving from the non-hiring phase

Proposition 5 brings a similar situation as optimal control problems with state variable inequality constraints (SVICs). A state constraint equivalent to (15) in SVICs would have worked as a binding constraint on controls only in its derivative form as $f' \dot{l} = \dot{y}$. By nature of the derivative form, it does not tell when the constraint becomes binding or off-binding, which requires additional information on the level. In the current problem, binding constraint (15) is not given but derived from optimal conditions, however the same property holds. It is derived from the truncatability of the problem into subperiods Λ^o and $(\Lambda^o)^c$ as do the jump conditions in SVICs. The truncated problem for initial time $0 \notin \Lambda^o$ and $E^\Lambda \neq \emptyset$ can be regarded as a problem with the terminal surface (15) for the entering time t^e and the terminal state l_{t^e} to be optimally determined. The problem brings the terminal costate variable to be $\lambda_{t^e} > 0$. Since $\lambda = 0$ in Λ^o, it implies costate discontinuity at entering time. The same holds for the leaving time.

Suppose $E^\Lambda \neq \emptyset$ and $0 \notin \Lambda^o$. Let $t^e \in E^\Lambda$ be the first entering time to Λ^o. Regarding t^e as the terminal time in discretion, the truncated problem as of time zero rewrites as follows.

$$(17) \qquad \Pi(0, l_0) = \max_{\hat{l}, x, t^e} \int_0^{t^e} \left(f(\hat{l}) - w \, l - c \right) e^{-rt} dt + \Pi(t^e, l_{t^e}) e^{-rt^e}$$

State transition and constraints remain the same and the terminal surface is given by

(18) $$f(l_{t^e}) = y_{t^e}.$$

Define the terminal-time Lagrangean $\Phi(t, l_t)$ by

$$\Phi(t, l_t) := \Pi(t, l_t) + \nu^e \left(y_t - f(l_t)\right)$$

where ν^e is the Lagrange multiplier attached to the terminal surface. Define general notations $z(T_-) := \lim_{t \uparrow T} z(t)$ and $z(T_+) := \lim_{t \downarrow T} z(t)$ for any time-dependent variable z. Then, the terminal condition on the costate variable becomes

$$\lambda(t^e_-) = \frac{\partial \Phi}{\partial l} = \frac{\partial \Pi}{\partial l} - \nu^e f'.$$

Since $\partial \Pi / \partial l = \lambda(t^e) = 0$, it implies

(19) $$\lambda(t^e_-) = -\nu^e f'\left(l(t^e)\right).$$

The condition on terminal time is given by $\mathscr{H}(t^e_-) - r\left[\Pi(t^e, l_{t^e}) + \nu\left(y_{t^e} - f(l_{t^e})\right)\right] = \mathscr{H}(t^e_+) + \nu^e \dot{y}_-$, which simplifies to

(20) $$f\left(\hat{l}(t^e_-)\right) - y + \lambda(t^e_-)\left[g(\tilde{l}(t^e_-)) - \sigma l\right] = r\Pi(t^e, l(t^e)) - \nu^e \dot{y}_-.$$

(19) and (20) settle the relation between the entering time and the costate variable as follows.

(21) $$\left(\dot{l}_-(\lambda_-) - \dot{l}_+\right)\lambda_- = y - f(\hat{l}_-(\lambda_-)) + r\Pi(t^e, l(t^e))$$

where $\dot{l}_-(\lambda_-) = g(\tilde{l}_-(\lambda_-)) - \sigma l$ and $\dot{l}_+ = \dot{y}/f'\left(l(t^e)\right)$. Note that $\dot{l}_-$ is a function of λ_-. This condition can be interpreted as $\partial \Pi / \partial t = r\Pi + (y - wl - c) + \dot{l}_+ \partial \Pi / \partial l$, i.e. the direct benefit of postponing the entering time equates the return of the firm's value plus instantaneous profit plus the increased value caused by change of employment.

PROPOSITION 6:   *If $t^e - \varepsilon \in W$ for any $\varepsilon > 0$, then $\lambda(t^e_-) > 0$.*

PROOF:
   Suppose $\lambda(t^e_-) = 0$. Then, $\tilde{l}(t^e_-) = 0$ and $\hat{l}(t^e_-) = l(t^e_-)$ from (6) and $\nu = 0$ from (19). Then, (20) becomes $f\left(l(t^e_-)\right) - y = r\Pi(t^e_-, l(t^e))$. However, since $l(t^e)$ is on the terminal surface, it implies $\Pi(t^e_-, l(t^e)) = 0$. Since the firm never operates with $\Pi(t, l(t)) < 0$ for any $t \geq t^e$, it means $\Pi(t, l(t)) \equiv 0$ for any $t \geq t^e$, which in turn implies

(22) $$f(\hat{l}_t) = w_t l_t + c_t$$

for any $t \geq t^e$. On the other hand, since $\lambda(t^e) = \partial\Pi/\partial l = 0$, $\Pi(t^e, l) = 0$ holds for any initial value l at the entering time. Applying similar argument as above, (22) holds for different initial values of labor at the entering time. Pick up $l(t^e) + \varepsilon$ for sufficiently small $\varepsilon > 0$. Since (22) holds for $l(t^e)$, $f(l(t^e) + \varepsilon) \neq w_t (l_t + \varepsilon) + c_t$ holds by strict concavity of f, which contradicts (22). Therefore, $\lambda(t^e_-) \neq 0$. However, Proposition 4 implies $\lambda(t^e_-) \geq 0$ if $t^e - \varepsilon \in W$. Thus, $\lambda(t^e_-) > 0$. $\qquad\square$

Since $\lambda(t^e_+) = 0$, Proposition 6 implies general discontinuity of λ at entering time. From (21), $\lambda(t^e_-) > 0$ and $f(\hat{l}(t^e_-)) < y$ imply $\dot{l}(t^e_-) > \dot{y}/f'(l(t^e))$ since $\Pi(t^e, l(t^e)) > 0$. It means that the path of l "bumps" into the demand surface as shown in Figure 4a. The effect of the same costate discontinuity is reflected in the diagram of production possibility set in Figure 4b. The optimal path goes into point A on the production frontier where shadow price of hiring λ is strictly positive at the entering time and jumps to point B. Since B is bound by the demand constraint as Proposition 5 predicts, point A must be unbound because $\hat{l}(t^e_-) < \hat{l}(t^e_+)$. The entering behavior is summarized in a literate manner

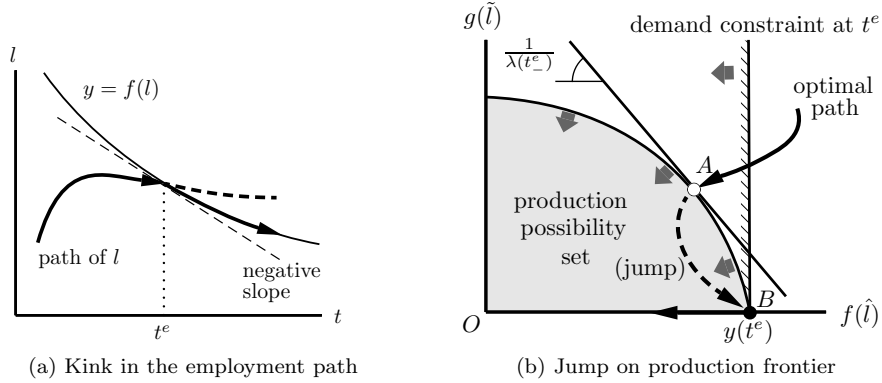(a) Kink in the employment path   (b) Jump on production frontier

Figure 4. : Entering behavior

as follows.

PROPOSITION 7: *At entering time, production discontinuously increases so that unbinding demand constraint beforehand becomes binding afterwards. Hiring discontinuously decreases from strictly positive to zero. Costate variable λ jumps from strictly positive to zero. Demand dual μ jumps from zero to strictly positive. Time path of l kinks so that $\dot{l}$ jumps downwards.*

Figure 5a draws how the jump condition (21) and the terminal surface (18) determine the first entering time. Starting from the initial employment l_0 and the hypothetical initial costate value λ_0, transitional equations (1) and (5) govern the dynamics. The entering time must satisfy the jump condition (21) which is

drawn as as a broken curve on the above plane. The time when the path of λ encounters the surface of the jump condition is the entering time. At this time, the production jumps from $f(\hat{l})$ to $f(l)$ and the latter coincides with y. It only holds for the correct initial costate value. If not, the initial hypothesis on λ must be corrected. For the rest of entering times, if exist, λ_0 and l_0 should be replaced by $\lambda(t^l_+)$ and $l(t^l)$ where t^l is the previous leaving time.
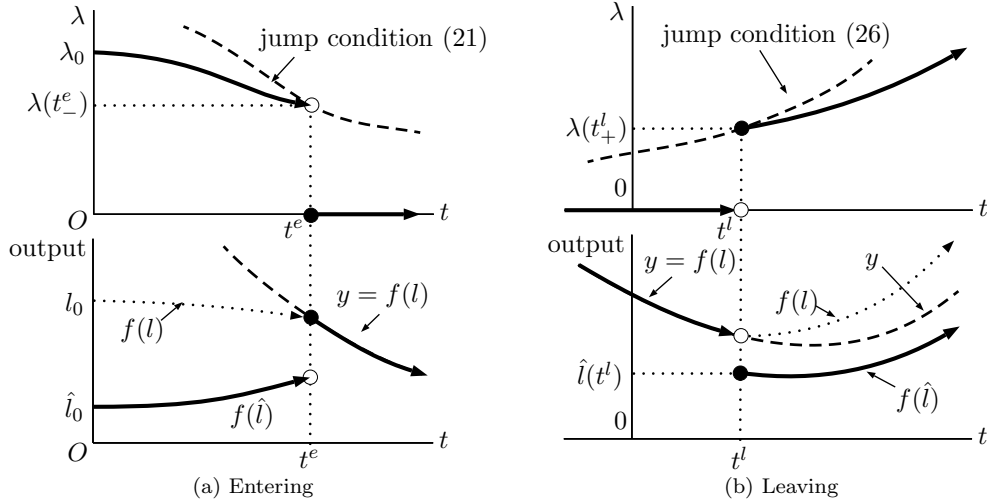

Figure 5. : Determining junction time

The leaving behavior is analyzed in a similar fashion. Now, consider a production decision as of t^e. Again, the problem is truncated at $t^l \in \bar{\Lambda}^o_{t^e}$ to check the differentiability at the interface of the firing phase. As we have already obtained the optimal policy on Λ^o in section IV, the remaining problem is the choice of t^l, which is obtained from the terminal condition. The problem is the same as (17) except that the initial and terminal time is replaced by t^e and t^l, respectively, and the discounting period is modified to the interval starting from t^e. So, after applying the optimal policy on Λ^o, we have

$$\Pi(t^e, l_{t^e}) = \max_{t^l} \int_{t^e}^{t^l} \left(y - wf^{-1}(y) - c \right) e^{-r(t-t^e)} dt + \Pi(t^l, l_{t^l}) e^{-r(t^l-t^e)}.$$

Transitions and constraints remain the same. At the leaving time, the following terminal constraint must hold.

(23) $$f(l_{t^l}) = y_{t^l}$$

This is required before optimization is undertaken, since Proposition 5 tells that

the effectiveness of the constraint (15) is derived from optimality only for $t \in \mathring{\Lambda}^o \cap W$ but it extends to $t \in \bar{\Lambda}^o \cap W$ by *continuity* of l, which is an assumption imposed on the maximum principle. However, one may find subtlety in inclusion of this terminal constraint in the problem. So let us check its validity. First, we proceed with the constraint. The terminal condition becomes

$$(24) \quad -\left[f\big(\hat{l}(t^l)\big) - f\big(l(t^l_-)\big) \right] - \lambda(t^l)\left[g(\tilde{l}(t^l) - \sigma l \right] = r\Pi\big(t^l, f^{-1}(y_{t^l})\big) - \nu^l \dot{y}$$

$$(25) \quad \lambda(t^l) = \nu^l f'\big(l(t^l)\big)$$

where ν^l is the Lagrange multiplier adjoint to the terminal constraint. If there were not a terminal condition, we can set $\nu^l = 0$ which implies that $\lambda(t^l)$ becomes continuous at zero. By imposing $\nu^l = \lambda(t^l) = 0$, we have $f\big(l(t^l_-)\big) - f\big(\hat{l}(t^l)\big) = r\Pi\big(t^l, f^{-1}(y_{t^l})\big)$. Since $\lambda_{t^l} = 0$, we have $\hat{l}(t^l) = l(t^l)$ implying $\Pi\big(t^l, f^{-1}(y_{t^l})\big) = 0$. This is impossible to happen as argued in the proof of Proposition 6. Therefore, the terminal constraint (23) is required for the economy to exist within the framework of the maximum principle. Since $\nu = 0$ is impossible in (25), it also showed the following.

PROPOSITION 8: λ *is discontinuous at the leaving time so that* $\lambda(t^l_-) = 0$ *and* $\lambda(t^l) > 0$ *hold.*

Next, derive the jump condition for leaving. From (24) and (25), we obtain

$$(26) \quad \left(\dot{l}_+(\lambda_+) - \dot{l}_- \right)\lambda_+ = y - f(\hat{l}_+)) - r\Pi(t^l, l(t^l)).$$

where $\dot{l}_+(\lambda_+) = g\big(\tilde{l}_+(\lambda_+)\big) - \sigma l$ and $\dot{l}_- = \dot{y}/f'\big(l(t^l)\big)$. Note that $\dot{l}(t^l_+)$ is a function of $\lambda(t^l_+)$. We can rewrite (26) as

$$y - wl + \dot{l}(t^l_-)\lambda(t^l_+) = r\Pi(t^l, l(t^l)) + f\big(\hat{l}(t^l_+)\big) - wl + \dot{l}(t^l_+)\lambda(t^l_+).$$

It can be interpreted that, on the leaving time, the benefits of postponed and immediate leave become equal. The left-hand side of the equation is the benefit of postponing the leave. By retarding the leave by dt, the firm receives bound instantaneous profits and the value of employment change according to bound dynamics. The right-hand side is the benefit of immediate leave. By obtaining the new state Π, the firm receives its return, and also instantaneous profits and the value of employment change both according to the unbound path.

   The leaving behavior is summarized symmetrically to the entering case.

PROPOSITION 9:   *At leaving time, production discontinuously decreases so that binding demand constraint beforehand becomes unbinding afterwards. Hiring discontinuously increases from zero to strictly positive. Costate variable λ jumps from zero to strictly positive. Demand dual μ jumps from strictly positive to zero. Time path of l kinks so that $\dot{l}$ jumps upwards.*

Figure 5b draws how the jump condition (26) and the terminal surface (18) determine the entering time. Different from ordinary optimization problems, the initial employment in the truncated problem after the leave is not given. The choice of t^l directly determines it according to the demand constraint (23). The jump condition (21) drawn as as a broken curve on the above plane simultaneously determines $\lambda(t^l_+)$ which corresponds to the hypothesis on λ_0. If the choice of t^l is correct, the dynamics of λ (5) provides the value of λ which satisfies the transversality condition if t^l is the last leaving time. If there is another entering to Λ^o, the dynamics satisfies the entering conditions described above at the next entering time. Figure 6 draws the kink in the employment path and the jump in the production frontier at the leaving time.
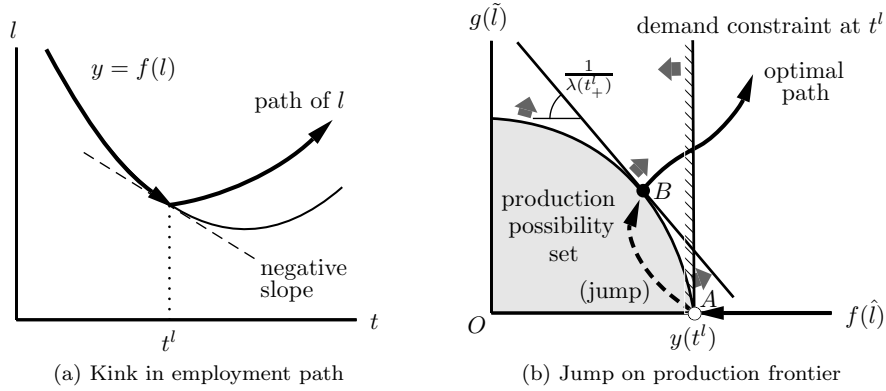

(a) Kink in employment path   (b) Jump on production frontier

Figure 6. : Leaving behavior

VI.   Weak labor hoarding

A labor asset model can have labor hoarding within business cycles since its demand depends on the value of the labor asset which does not necessarily meet the demand for the spot labor expense. In the present model with no firing cost, it only occurs in a weak sense that firing will not take place even when the decline in the spot labor demand is more than the natural separation. In this case, since the value of labor is positive, they are instead hired in the hiring sector. This is easily observed by an example. Figure 7 shows a periodic steady state of a toy economy in which the demand has period 2π and the level of demand falls below the unconstrained steady state only in $t \in (0.61, 2.53)$ in the principal domain $[0, 2\pi)$.[11] Studying the properties of periodic steady state is beneficial

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[11]Note that this is a non-autonomous dynamical system. The periodicity is brought by that of the demand constraint potentially binding. The toy economy has the production function $f(\hat{l}) = 5\,\hat{l}^{3/4}$ and
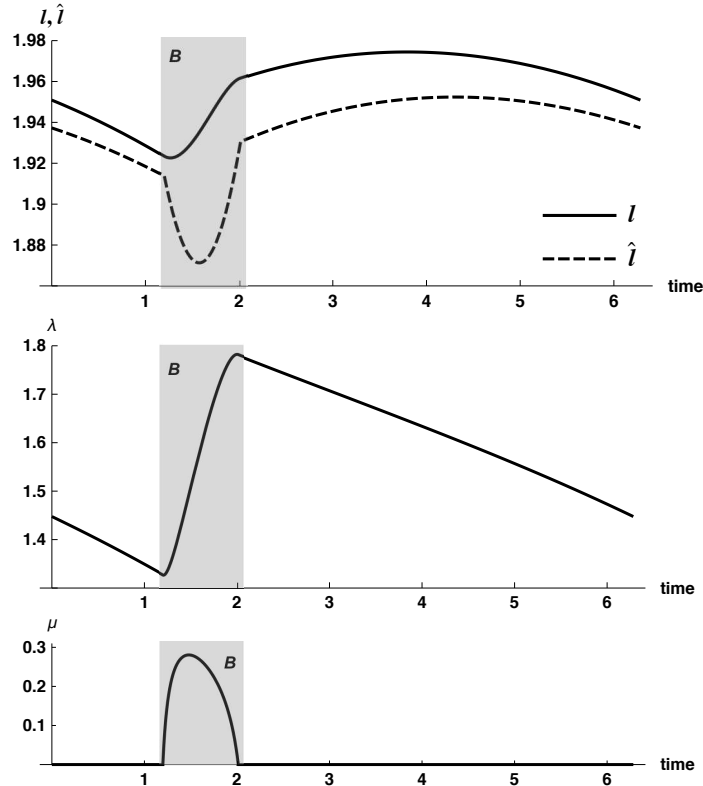
Figure 7. : Periodic steady state of the toy economy

to close this type of optimization problem with infinite horizon since specifying external force y for infinite period is literally impossible and effects of faraway future is discounted anyway. Instead of keeping y open-ended, we can safely close it as a loop with sufficiently long period.[12] The optimal path binds to the constraint only in its subset, i.e. $t \in (1.05, 2.07) =: B$. As shown in the first figure, although $\hat{l} = f^{-1}(y)$ in B, employment begins to increase right after entering in B. Redundant labor is utilized to enforce the hiring sector in preparation for future increase of production. In B^c, the firm *unboundedly chooses* to operate in a lower production level than the overall unconstrained case in which the optimal employment is constant at $l = 2.17$. Note that, in the constraint binding phase, the increase of employment is driven by the improvement of labor value shown in the second figure. Since $\dot{\lambda} = (r+\sigma)\lambda - (1-\mu)f'(\hat{l}) + w$, the rise of μ in the binding

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$g(\tilde{l}) = \tilde{l}^{3/4}$. Separation occurs at the rate 0.03. Discount rate is set to 0.05. The demand constraint is $y = 10 + 2\sin(t - \pi)$ and wage rate is constant at $w = 3$. Note that, even though demand constraint is binding only in some subperiod, it brings periodicity upon the whole optimal path. For the properties of periodic forced oscillation for linearized systems, see Kato, Naito and Shin (2005).

[12]This is true even when there is a long-run trend in y.

phase more quickly improves the value of labor λ by adding external forces. Note that, in the unbinding phase, the external force that affects λ is only through the change of the marginal productivity of labor. Drastic improvement of labor value happens more easily in the binding phase in this sense. Also, note that the existence of a small period of the binding phase can affect the whole dynamics. In this example, the binding phase occupies only sixteen percent of the total period.

VII.   Strong labor hoarding and firing cost

On the other hand, if there exists firing cost, strong labor hoarding can arise in the sense that part of employment is put idle. Assume that there exists convex firing cost $\kappa(x)$ where $\kappa : \mathbb{R}_+ \to \mathbb{R}$, $\kappa', \kappa'' > 0$, $\kappa(0) = \kappa'(0) = 0$ and $\kappa'(+\infty) = +\infty$. This specification implicitly assumes that firing activity does not consume internal human resources. This would be approximately true if sufficient information on worker properties that is necessary for selection of firing target is already accumulated within everyday work, and if the main cost of firing is pecuniary compensation. We also impose a moderate assumption that $f'(\hat{l}) \geq w$ reflecting the bargaining outcome that the value of profits is strictly positive. We modify the objective function to $\Pi = \max_{\hat{l},x} \int_0^\infty \left(f(\hat{l}) - w\,l - \kappa(x) - c \right) \mathrm{e}^{-rt} \mathrm{d}t$, the labor transition to $\dot{l} = g(\tilde{l}) - \sigma l - x$ where $\tilde{l}$ is the employment in the hiring sector and add conditions $\tilde{l} \geq 0$ and $\hat{l} + \tilde{l} \leq l$ to allow for idle employment. Denote adjoint variables to the last two constraints by η and θ, respectively. The demand constraint is unchanged. Also, we can safely omit the constraint $\hat{l} \geq 0$ as far as $y > 0$ holds. Then, the first order conditions become

$$(27) \qquad\qquad (1 - \mu)\, f'(\hat{l}) = g'(\tilde{l})\, \lambda + \eta = \theta$$

$$(28) \qquad\qquad x = \begin{cases} 0 & \text{if } \lambda \geq 0 \\ \kappa'^{-1}(-\lambda) & \text{if } \lambda \leq 0. \end{cases}$$

Costate dynamics is unchanged from (5). Different from the previous model, $\lambda < 0$ is required for firing to exist. Suppose $\lambda < 0$ so that $x > 0$. From (27) and $\mu \geq 0$, we get $\eta > 0$ and $\tilde{l} = 0$. If $\mu = 0$, (8) implies $\dot{\lambda} < 0$. If $\mu = 0$ continues to hold, it finally violates the constraint $l \geq 0$. So, there must be a period that $\mu > 0$ holds until $\lambda \geq 0$ is achieved. Take such a period. Since $\mu > 0$, it requires $\dot{l} = \dot{y}/f'(f^{-1}(y))$ as far as we assume $\hat{l} = l$, which is generically impossible for a general function y. So, $\mu = 1 > 0$, $\theta = 0$ and thus the labor hoarding relation $\hat{l} = f^{-1}(y) \leq l$ generically holds.

VIII.   Implication to linear firing cost

The firing cost in the previous section is specified an exogenous factor represented by $\kappa(x)$. Such specification minimizes alteration of the basic model but may blur the actual origin of the cost. If the origin is viewed as that of human

resources directed to firing activities, it would be natural to assume $\kappa(\cdot)$ to be convex with the same reason as the hiring cost. A formal specification for such firing cost should extend the basic model to add the firing sector. However, empirical studies such as Kramarz and Michaud (2010) show that there is quite different firing cost structure among different countries, implying that empirical total firing cost may need to keep functional form of $\kappa(\cdot)$ more general. Kramarz and Michaud (2010) points out firing regulations in France and finds linear firing cost from French data. If firing cost arising from the human resources is negligible, the total firing cost may be indeed almost linear. In such a case, the singularity results of the basic model apply, changing the definition of Λ^o being intervals of $\lambda = -k$ if $\kappa(x) = kx$. It assumes that firing cost does not accompany fixed cost. Then, the optimal condition for firing (7) changes to

$$
x = \begin{cases} 0 & \text{if } \lambda > -k \\ [0, \bar{x}] & \text{if } \lambda = -k \\ \bar{x} & \text{if } \lambda < -k \end{cases}
$$

for arbitrarily large $\bar{x}$, whereas other conditions are unchanged. It implies that there exists no-firing and no-hiring interval in λ, i.e. $-k < \lambda \leq 0$. In that interval, $\eta > 0$ and $\theta \geq 0$ hold in (27). The same arguments in the previous section apply and in general strong labor hoarding is observed in that interval.

IX. Wages

Based on the optimal behavior of the firm derived above, one can investigate the implication to the bargained wages. Let employee's value be denoted by E and that of the unemployed by U. Matching rate for the unemployed is μ. Unemployment benefit is b. Then, the dynamics of E and U satisfy

$$
rE_t = w_t + \sigma_t (U_t - E_t) + \dot{E}_t
$$
$$
rU_t = b_t + \mu_t (E_t - U_t) + \dot{U}_t
$$

which gives the relation for employee's premium in value

(29) $$\dot{E}_t - \dot{U}_t = (r + \mu_t + \sigma_t)(E_t - U_t) - (w_t - b_t).$$

Assume that employment contract is made in each moment before production activity is undertaken and the wage rate is filled in the contract. The wage bargaining scene is quite different between the firing and non-firing phases. In the firing phase, they face a negotiation about who should be eliminated from the coalition in the firm. The situation is naturally noncooperative. Since the number to hire is already settled according to the optimal hiring plan of the firm, the task of the labor-management negotiation is to select the ones to hire. The situation can be drawn as a one-shot noncooperative game where each worker offers wages

he/she is willing to accept and the firm decides take or leave for each worker. The total number the firm hires is strictly less than the number of current employees. The payoff of workers is E if employed and U if unemployed. The payoff of the firm is the negative of the employment cost which is the sum (integral) of wages for all employed workers, $-\int_0^l w(i)\,\mathrm{d}i$ where i is an index of workers and $w(i)$ is limited to a measurable function.[13] Then, Nash equilibrium is characterized by $E(w) = U$. Namely, in the firing phase, $E_t = U_t$ for any $t \in \Lambda^o$ holds which implies $\dot{E}_t = \dot{U}_t$ for any $t \in X \subset \Lambda^o$. It implies

$$(30) \qquad\qquad\qquad\qquad\qquad\qquad w_t = b_t$$

for any $t \in X$ from (29).

On the other hand, wage bargaining in the non-firing phase is cooperative. The wage rate here is assumed to be settled bilaterally via generalized Nash bargaining solution. Namely, wage function maximizes generalized Nash product

$$(31) \qquad\qquad\qquad\qquad \mathscr{N} := \lambda^\beta \left(E - U\right)^{1-\beta}$$

where $\beta \in [0, 1]$ is a parameter representing the relative bargaining power between the firm and a worker. Given that cooperation collapses at the entering time to the firing phase, value of players is evaluated up to the next entering time. (8) implies that the costate variable of the firm becomes

$$\lambda_t = \int_t^{t^e} \left[(1 - \mu_\xi)\, f'(\hat{l}_\xi) - w_\xi\right] \mathrm{e}^{-\int_t^\xi (r+\sigma_s)\mathrm{d}s}\mathrm{d}\xi + C$$

where C is an integration constant. It implies the derivative of the costate variable becomes

$$(32) \qquad\qquad\qquad \mathrm{d}(\lambda_t) = -\int_t^{t^e} \delta w_\xi\, \mathrm{e}^{-\int_t^\xi (r+\sigma_s)\mathrm{d}s}\mathrm{d}\xi$$

for a variation of a wage function δw. On the other hand, the value premium of employees is obtained from (29), i.e.

$$E_t - U_t = \int_t^{t^e} \left(w_\xi - b_\xi\right)\mathrm{e}^{-\int_t^\xi (r+\mu_s+\sigma_s)\mathrm{d}s}\mathrm{d}\xi + C.$$

It gives the derivative of the value premium

$$(33) \qquad\qquad\qquad \mathrm{d}\left(E_t - U_t\right) = \int_t^{t^e} \delta w_\xi\, \mathrm{e}^{-\int_t^\xi (r+\mu_s+\sigma_s)\mathrm{d}s}\mathrm{d}\xi$$

---

[13]In other words, this is a limit of a finite game and the equilibrium is confined to the limit of equilibria in those finite games.

for a variation of a wage function δw. Then, the derivative of (31) is given by

$$d\mathscr{N} = \lambda_t^{\beta-1}(E_t - U_t)^{-\beta} \times$$
$$\int_t^{t^e}\left[(1-\beta)\,\lambda_\xi\,\mathrm{e}^{-\int_t^\xi(r+\mu_s+\sigma_s)\mathrm{d}s} - \beta\,(E_\xi - U_\xi)\,\mathrm{e}^{-\int_t^\xi(r+\sigma_s)\mathrm{d}s}\right]\delta w_\xi\mathrm{d}\xi$$

using (32) and (33). The coefficient of δw is zero at extremum, which implies

$$\beta\,(E_\xi - U_\xi) = (1-\beta)\,\lambda_\xi\,\mathrm{e}^{-\int_t^\xi\mu_s\mathrm{d}s}.$$

Using (32) and (33) again, the above relation becomes

$$\int_t^{t^e} w_\xi\,\mathrm{e}^{-\int_t^\xi(r+\mu_s+\sigma_s)\mathrm{d}s}\mathrm{d}\xi$$
$$= \int_t^{t^e}\left[\beta b_\xi + (1-\beta)\,(1-\mu_\xi)\,f'(\hat{l}_\xi)\right]\mathrm{e}^{-\int_t^\xi(r+\mu_s+\sigma_s)\mathrm{d}s}\mathrm{d}\xi,$$

which must hold for *any* $t \in (\Lambda^o)^c$. Thus, differentiating by t obtains

(34) $$w_t = \beta b_t + (1-\beta)\,(1-\mu_t)\,f'(\hat{l}_t).$$

Wage equation (34) has some novel implications. First, when the demand constraint is binding, lower the income, lower the wage rate at steady states. This is the direct result of Proposition 2. It is because even though the marginal productivity of labor (MPL) rises, the effective MPL $(1-\mu_t)\,f'(\hat{l}_t)$ falls due to the rise of μ_t. Second, even though the demand constraint is not binding, if it is expected to be binding in future, the current wage rate is lowered. This is because lowered λ causes the rise in the share of employment in the production sector for given total employment, which leads to the decrease of MPL. Third, (30) and (34) imply that wage rate becomes discontinuous at junction times. At entering time to the firing phase, the wage rate discontinuously falls to the same level as the unemployment benefit. Leaving time from the firing phase shows a symmetric move.

The main model in the present paper adopted exogenous wage bargaining. However, the result obtained here can be used to endogenize it. Wage equation (34) implies that more employment in the production sector tends to lower the wage rate, which the firm now takes into account. So, the first order condition (6) becomes

(6') $$(1-\mu)\left[f'(\hat{l}) - (1-\beta)\,f''(\hat{l})\,l\right] = \lambda\,g'(l-\hat{l}) + \eta$$

which has an effect to raise the share of employment in the production sector than the original model.

X. Conclusion

The represented model may seem to have imposed too much distortions on a pure competitive model. However, the basic assumptions introduced as distortion are only existence of variety both in goods and workers. Other conditions are derived from there. Whereas variety of goods are observable, that of workers are not by nature of human ability which causes search in face of hiring. Therefore, their treatment is not symmetric. The convex cost of hiring rises naturally from decreasing returns of individual hiring activity, which prohibits jumps to the steady state. It implies that the "transition" process matters. Moreover, sufficiently high degree of differentiation in output goods does not always allow simple transition to the unbounded steady state. Rather it confers a main role of determination of the whole path on coordinated expectation. In those cases, the transition process is not a transition any more.

Viewing the problem from the perspective of the maximum principle, the present model showed that the jump of costate variables can occur *only with control constraints* in singular problems. Jump condition of the state-constrained problems have been well-known. However, singularity can bring indifferentiability of state variables on boundary even when only control constraints are included in the problem, which truncates the problem into subperiods in a similar fashion to the state-constrained problems and brings about the costate jumps.

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