Priority and Egalitarian Allocation in the Capability Approach

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Abstract

Individuals have different capability of transforming resources into basic human functionings. Our priority principles roughly say that the more disabled a person is, the greater access to resources she should be provided. Extending Moreno-Ternero and Roemer (2006, "Impartiality, priority and solidarity in the theory of justice," Econometrica 74, 1419-1427) and Chun, Jang, Ju (2014, "Priority, Solidarity, and Egalitarianism," Social Choice and Welfare, 43, 577-589) to multidimensional setting, we provide characterization of egalitarian allocation rules using our priority axioms and other standard axioms in the literature of fair allocation. Our egalitarian allocation rules choose allocations where all persons achieve the same level of capability index (the index function aggregate resources and basic human functionings into a real number representing the level of capability). Among these rules are resource and output egalitarian rules. The output egalitarian rule adopting the human development index is a central example in the family. Keywords: priority, egalitarianism, capability approach, solidarity

1 Introduction

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2 Model

Consider a society with a finite number of sectors and a finite number of agents. The society allocates resources to the agents and individual outputs are interpersonally comparable. Each individual, after receiving resource from the society, decides how much to assign into multiple sectors, and then produces multiple-dimensional outputs. Assume that the same kind of outputs are interpersonally comparable, but outputs of different kinds are not comparable.

Let $N = \{1, 2, \cdots, n\}$ ($n \geq 3$) be a set of agents, $M = \{1, 2, \cdots, m\}$ ($m \geq 2$) be the set of sectors, and $W \in \mathbb{R}_+$ be the total available resource. Each agent $i \in N$ is characterized by a profile of division functions $\gamma_i = (\gamma_{1i}, \cdots, \gamma_{mi})$ and output functions $y_i = (y_{1i}, \cdots, y_{mi})$, where $\gamma_i : \mathbb{R}_+ \to \mathbb{R}_+^m$ is assumed to satisfy

(i) (efficiency) $\gamma_i(W_i) > 0$[1] for any $W_i > 0$ and $\sum_{t \in M} \gamma_{ti}(W_i) = W_i$,

(ii) (*sector monotonicity*) $\gamma_i(W_i) > \gamma_i(\hat{W}_i)$ for any $W_i > \hat{W}_i$,[2]

(iii) (sector unboundedness) $\gamma_{ti}(W_i) \to \infty$ as $W_i \to \infty$ for each t.[3]

and $y_{ti} : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be continuous, strictly increasing, unbounded, and $y_{ti}(0) = 0$ for each $t \in M$. Let Γ be the set of all such division functions and $\mathcal{Y}^*$ be the set of all such output functions. An economy $e = (\gamma, y, W)$ consists of a profile of agents' division functions $\gamma \equiv (\gamma_i) \in \Gamma^n$,

[1]For any vectors $x = (x_1, \cdots, x_n)$ and $x' = (x'_1, \cdots, x'_n)$, we denote $x' > x$ if $x'_k > x_k$ for all $k = 1, \cdots, n$, and $x' \geq x$ if $x'_k \geq x_k$ for all $k = 1, \cdots, n$.

[2]If an individual gets more resource from the society, she assigns more resource into both sectors.

[3]Each individual's decision plan for both sectors are unbounded.

a profile of agents' output functions $y \equiv (y_{1i}, \cdots, y_{mi})_{i \in N} \in (\mathcal{Y}^*)^{mn}$, and the available resource $W \in \mathbb{R}_+$. Let $\mathcal{E}^* \equiv \Gamma^n \times (\mathcal{Y}^*)^{mn} \times \mathbb{R}_+$ be the set of all economies, the *universal domain.* Domain $\mathcal{E}$ is a *covering domain* so that the graphs of division functions in Γ and output functions in $\mathcal{Y}^m$ cover the positive quadrant, that is, for all $(w_1, \cdots, w_m) \in \mathbb{R}^m_{++}$ and $a, b \in \mathbb{R}_{++}$, there is $\gamma_i \in \Gamma$ and $y_{ti} \in \mathcal{Y}$ such that $\gamma(a) = (w_1, \cdots, w_m)$ and $y_{ti}(a) = b$. (We suppress * in $\mathcal{E}^*$ and $\mathcal{Y}^*$ when there is no confusion.)

An *allocation rule* $F \colon \mathcal{E} \to \mathbb{R}^{mn}_+$ associates with each economy $e = (\gamma, y, W) \in \mathcal{E}$ an allocation of individual resources for producing outputs in each sector, $F(e) = (F_i(e))_{\{i \in N\}} = ((F_{ti}(e))_{\{t \in M\}})_{\{i \in N\}} \in \mathbb{R}^{mn}_+$ satisfying the *resource constraint*:

$$\sum_{t \in M, i \in N} F_{ti}(e) = W \text{ and } \gamma_i\Big(\sum_{t \in M} F_{ti}(e)\Big) = F_i(e) \; \forall i \in N,$$

division function invariance: for each $i \in N$,

$$F_i(e) = F_i(\gamma', y, W) \text{ if } \gamma_i\Big(\sum_{t \in M} F_{ti}(e)\Big) = \gamma'_i\Big(\sum_{t \in M} F_{ti}(e)\Big),$$

and *sector unboundedness*: for each $i \in N$ and $t \in M$,

$$\lim_{W \to \infty} F_{ti}(\gamma, y, W) \to \infty.$$

We denote $S_i(e) = \sum_{t \in M} F_{ti}(e)$ and $y_i(F_i(e)) = (y_{1i}(F_{1i}(e)), \cdots, y_{mi}(F_{mi}(e)))$ for notational convenience. Each kind of output is produced by the same kind of resources. For example, $y_{1i}(F_{1i}(e))$ is agent i's first output with her share of the first resources. According to the resource constraint, we can consider that each individual, after receiving its individual resource, has its own division plan that the social rule cannot control so that the rule takes the division function profile into consideration when determining the distribution.

3 Axioms

3.1 Priority axioms

No-Domination. For all $e = (\gamma, y, W) \in \mathcal{E}$, there is no pair $i, j \in N$ such that $(F_i(e), y_i(F_i(e))) \leq (F_j(e), y_j(F_j(e)))$ and $(F_{ti}(e), y_{ti}(F_{ti}(e))) < (F_{tj}(e), y_{ti}(F_{tj}(e)))$ for some $t \in M$.

For all $i, j \in N, t \in M$, and all $y_{ti}, y_{tj} \in \mathcal{Y}$, denote $y_{ti} \leq y_{tj}$ if $y_i(x) \leq y_j(x)$ for all $x \in \mathbb{R}_+$, and $y_{ti} < y_{tj}$ if $y_i(x) < y_j(x)$ for all $x \in \mathbb{R}_+$. Moreover, for all $i, j \in N$ and all $y_i, y_j \in \mathcal{Y}^n$, we say y_i is ***disabled relative to*** y_j if $y_i(w) \leq y_j(w)$ for all $w \in \mathbb{R}^m_+$ to denote as $y_i(w) \leq y_j(w)$, and y_i is ***strictly disabled relative to*** y_j if $y_i(w) < y_j(w)$ for all $w \in \mathbb{R}^m_+$ to denote as $y_i(w) < y_j(w)$.

Order-Preservation. For all $e = (\gamma, y, W) \in \mathcal{E}$ and all $i, j \in N$, if $y_i \leq y_j$, then $F_i(e) \geq F_j(e)$.

No-Reversal (in Outputs). For all $e = (\gamma, y, W) \in \mathcal{E}$ and all $i, j \in N$, if $y_i \leq y_j$, then $y_i(F_i(e)) \leq y_j(F_j(e))$.

Note that each of the three axioms implies *equal treatment of equals*; for all $e = (\gamma, y, W) \in \mathcal{E}$, if $(\gamma_i, y_i) = (\gamma_j, y_j)$, then $F_i(e) = F_j(e)$. Note also that *no-domination* implies *order-preservation* and *no-reversal.*

Disability Monotonicity. For all $(\gamma, y, W) \in \mathcal{E}$, all $i \in N$, and all $y_i, y'_i \in \mathcal{Y}^n$, if $y'_i \leq y_i$, then $F_i(\gamma, (y'_i, y_{-i}), W) \geq F_i(\gamma, y, W)$.

3.2 Solidarity Axioms

Agreement. For all $e = (\gamma, y, W), e = (\gamma', y', W') \in \mathcal{E}$, and all $M \subseteq N$, if $(\gamma_M, y_M) = (\gamma'_M, y'_M)$, then either $F_M(e) = F_M(e')$, $F_M(e) > F_M(e')$, or $F_M(e) < F_M(e')$.

Separability. For all $e = (\gamma, y, W) \in \mathcal{E}, e' = (\gamma', y', W') \in \mathcal{E}$, and all $M \subset N$ such that $(\gamma_M, y_M) = (\gamma'_M, y'_M)$, if $\sum_{i \in M}(S_i(e)) = \sum_{i \in M}(S_i(e'))$, then $F_M(e) = F_M(e')$.

Separability is implied by *agreement*. It says that when the sum of resources remains unchanged for the unaffected agents, their allocation should be the same as before.

Resource Monotonicity. Let $e = (\gamma, y, W), e' = (\gamma, y, W') \in \mathcal{E}$ be such that $W' > W$. Then $F(e') > F(e)$.

Another implication of *agreement* is this axiom. When nothing but the amount of resources changes by a positive or negative shock, all agents should share its effect, that is, the amount of resources in each skill given to all agents should move in the same direction. The following axiom is also induced by *agreement*.

Resource Continuity. For all $y \in \mathcal{Y}^{mn}$, if a sequence of resources $\{W^n\}_{n \in \mathbb{N}}$ converges to W, then $\{F_{ti}(\gamma, y, W^n)\}_{n \in \mathbb{N}}$ converges to $F_{ti}(\gamma, y, W)$ for all $t \in M, i \in N$.

Evidently, an implication of *resource monotonicity* is *resource continuity*, that is, for all $(\gamma, y) \in \Gamma \times \mathcal{Y}^n$, if a sequence of resources $(W^n : n \in \mathbb{N})$ converges to W, then $(F(\gamma, y, W^n) : n \in \mathbb{N})$ converges to $F(\gamma, y, W)$.

4 Main Results

We first show that *agreement* is equivalent to the combination of *separability* and *resource monotonicity.*

Proposition 1. *A rule satisfies agreement if and only if it satisfies separability and resource monotonicity.*

Proof. The proof is an adaptation of Proposition 1 in Chun, Jang, and Ju (2014). □

Proposition 2. *If a rule satisfies no-reversal, disability monotonicity, and agreement, then it satisfies no-domination.*

Proof. Let F be a rule satisfying *agreement*, *no-reversal*, and *disability monotonicity.*

Step 1. For all $e = (\gamma, y, W) \in \mathcal{E}$, $i \in N$, and $y'_i \leq y_i$, $F_i(\gamma, (y'_i, y_{-i}), W) \geq F_i(e)$ and $F_{N\setminus\{i\}}(\gamma, (y'_i, y_{-i}), W) \leq F_{N\setminus\{i\}}(e)$.

Let any $e = (\gamma, y, W), e' = (\gamma, (y'_i, y_{-i}), W) \in \mathcal{E}$ with $y'_i \leq y_i$. By *disability monotonicity* and sector monotonicity, $F_i(e') \geq F_i(e)$, which also indicates that $S_i(e') \geq S_i(e)$. Therefore $\sum_{j\in N\setminus\{i\}} S_j(e') = W - S_i(e') \leq W - S_i(e) = \sum_{j\in N\setminus\{i\}} S_j(e)$. Finally, by *agreement*, $F_{N\setminus\{i\}}(e') \leq F_{N\setminus\{i\}}(e)$.

Step 2. F satisfies *no-domination*.

Suppose conversely that there exists $e = (\gamma, y, W)$ and $i, j \in N$ such that $(F_i(e), y_i(F_i(e))) \leq (F_j(e), y_j(F_j(e)))$ and $(F_{ti}(e), y_{ti}(F_{ti}(e))) < (F_{tj}(e), y_{ti}(F_{tj}(e)))$ for some $t \in M$. Let $y'_i \in \mathcal{Y}^n$ such that $y'_{si} \geq \max\{y_{si}, y_{sj}\}$ and $y'_{si}(F_{si}(e)) \leq y_{sj}(F_{sj}(e))$ for each $s \in M$, and $y'_{ti}(F_{ti}(e)) < y_{tj}(F_{tj}(e))$. Notice that both y_i and y_j are disabled relative to y'_i. Let $e' = (\gamma, (y'_i; y_{-i}), W)$. By Step 1, $F_i(e') \leq F_i(e)$ and $F_j(e') \geq F_j(e)$. Then $y'_{si}(F_{si}(e')) \leq y'_{si}(F_{si}(e)) \leq y_{sj}(F_{sj}(e)) \leq y_{sj}(F_{sj}(e'))$ for all $s \in M$ and $y'_{ti}(F_{ti}(e')) \leq y'_{ti}(F_{ti}(e)) < y_{tj}(F_{tj}(e)) \leq y_{tj}(F_{tj}(e'))$. That is, $y'_{si}(F_{si}(e')) \leq y_{sj}(F_{sj}(e'))$ with $y'_{ti}(F_{ti}(e')) < y_{tj}(F_{tj}(e'))$, which contradicts *no-reversal* at e'.

□

4.1 Index-egalitarianism

We first define a family of rules that satisfy *no-domination* and *agreement*. Let Φ be the class of all functions $\varphi : \mathbb{R}^{2m}_{++} \cup \{(0, 0, \cdots, 0)\} \to \mathbb{R}_+$, continuous on its domain and nondecreasing, satisfying $\varphi(0, 0, \cdots, 0) = 0$ and the following monotonicity property: for all $a = (a_t)_{t \in M}, a' = (a'_t)_{t \in M}, b = (b_t)_{t \in M}$, and $b' = (b'_t)_{t \in M} \in \mathbb{R}^m_+$,

- $\varphi(a', b') \geq \varphi(a, b)$ if $(a', b') \geq (a, b)$,
- $\varphi(a', b') > \varphi(a, b)$ if $(a', b') \geq (a, b)$ with $(a'_t, b'_t) > (a_t, b_t)$ for some t.

Given an economy $e = (\gamma, y, W)$, for each $i \in N$, let $\psi_i : \mathbb{R}^m_+ \to \mathbb{R}_+$ be such that for all $a = (a_t)_{t \in M} \in \mathbb{R}^m_+$, $\psi_i(a) = \varphi(a, (y_{ti}(a_t))_{t \in M})$. Since agents have different output functions, each of them faces a different index function. Notice that ψ_i is non-decreasing in each resources because output functions are strictly increasing, and that $\psi_i(\gamma_{1i}(W), \cdots, \gamma_{mi}(W))$ is strictly increasing in W. Also notice that the properties of φ imply continuity and the following monotonicity for ψ_i for all $i \in N$: for $a = (a_t)_{t\in M}, a' = (a'_t)_{t\in M} \in \mathbb{R}^m_+$,

(i) $\psi_i(a') \geq \psi_i(a)$ if $a' \geq a$, and

(ii) $\psi_i(a') > \psi_i(a)$ if $a' \geq a$ with $a'_t > a_t$ for some $t \in M$.

We now define the index-egalitarian rule (Moreno-Ternero and Roemer (2006)) that equalizes the φ-value for all the agents.

Index-Egalitarian Rule E^φ: For all $e \in \mathcal{E}$ and all $i \in N$, $E^\varphi_i = a_i = (a_{1i}, \cdots, a_{mi}) \in \mathbb{R}^m_+$, where $(a_i)_{i\in N}$ is chosen so that $\psi_1(a_1) = \psi_2(a_2) = \cdots = \psi_n(a_n)$.

A φ-index egalitarian rule allocates social endowment W into a φ-index egalitarian allocation $(a_{1i}, \cdots, a_{mi})_{i\in N}$ so that $\sum_{t\in M, i\in N} a_{ti} = W$. Note that for each given $\varphi \in \Phi$, E^φ is well-defined: the existence of E^φ is implied by monotonicity and continuity of ψ_i for each $i \in N$, and uniqueness can be easily shown.[4]

Theorem 1. *Given a domain $\mathcal{E}^*$, a rule satisfies no-domination and agreement if and only if it is index-egalitarian.*

The proof is provided in the appendix.

Theorem 2. *Given a domain $\mathcal{E}^*$, a rule satisfies no-reversal, disability monotonicity, and agreement if and only if it is index-egalitarian.*

[4]Suppose conversely that for some $e = (\gamma, y, W) \in \mathcal{E}$ there exist $a = (a_i)_{\{i\in N\}}, a' = (a'_i)_{\{i\in N\}} \in \mathbb{R}^{mn}_+$ such that $a \neq a'$ and $a, a' \in E^\varphi(e)$. Then, there exists $i \in N$ such that for some $t \in M$, either $a_{ti} > a'_{ti}$ or $a_{ti} < a'_{ti}$. Without loss of generality, assume that $a'_{1i} > a_{1i}$. By *sector monotonicity* of γ, $a'_{ti} > a_{ti}$ for all $t \in M$, which implies that $a'_i > a_i$ and therefore $\psi_i(a'_i) > \psi_i(a_i)$. Since both a' and a are φ-value equalizers, $\psi_j(a'_j) < \psi_j(a_j)$ for all $j \in N$, and therefore $\sum_{t\in M, j\in N}(a'_{tj}) > \sum_{t\in M, j\in N}(a_{tj})$, which contradicts the fact that $\sum_{j\in N, t\in M}(a'_{tj}) > \sum_{j\in N, t\in M}(a_{tj}) = W$.

Proof. The "only-if" part follows from Theorem **??** and Proposition **??**. It suffices to prove that all index-egalitarian rules satisfy *disability monotonicity.* Let $F = E^{\varphi}$ for some $\varphi \in \Phi$, $e = (\gamma, y, W) \in \mathcal{E}$, $i \in N$, $y'_i \in \mathcal{Y}$ be such that $y'_i \leq y_i$ and $e' = (\gamma, y', W)$ with $y' \equiv (y'_i, y_{-i})$. Then there exist $\lambda, \lambda' \geq 0$ such that for all $j \in N$, $\varphi(F_j(e), y_j(F_j(e))) = \lambda$ and $\varphi(F_j(e'), y'_j(F_j(e'))) = \lambda'$. First, suppose that $\lambda' > \lambda$. Then for all $j \in N \backslash \{i\}$, $\varphi(F_j(e), y_j(F_j(e))) < \varphi(F_j(e'), y'_j(F_j(e')))$, which implies that $F_j(e) < F_j(e')$. In the case of i, $\varphi(F_i(e), y_i(F_i(e))) < \varphi(F_i(e'), y'_i(F_i(e'))) \leq \varphi(F_i(e'), y_i(F_i(e')))$, which implies that $F_i(e) < F_i(e')$. Altogether, $W = \sum_{j \in N} F_j(e) < \sum_{j \in N} F_j(e') = W$, which is a contradiction. Therefore, $\lambda' \leq \lambda$. Then for all $j \neq i$, $\varphi(F_j(e), y_j(F_j(e))) \geq \varphi(F_j(e'), y_j(F_j(e')))$, which implies that $F_j(e) \geq F_j(e')$. And by *resource constraint,* $F_i(e) \leq F_i(e')$, as required by *disability monotonicity.*

□

4.2 Resource-index egalitarianism and output-index egalitarianism

We define two refinements of the family of index. Let $\mathcal{G}$ be the class of all functions $g : \mathbb{R}^m_{++} \cup \{(0, 0, \cdots, 0)\} \rightarrow \mathbb{R}_+$, continuous on its domain and nondecreasing, satisfying $f(0, \cdots, 0) = 0$ and the monotonicity property[5]. For any $a, b \in \mathbb{R}^m_{++} \cup \{(0, 0, \cdots, 0)\}$, we call that an index $\varphi \in \Phi$ is a *resource-index* if $\varphi(a, b) = g(a)$ for some $g \in \mathcal{G}$, and an *output-index* if $\varphi(a, b) = g(b)$ for some $g \in \mathcal{G}$. Let Φ^R be the class of all resource-index and Φ^O be the class of all output-index. A rule F is ***resource-index-egalitarian*** if it is a φ-index egalitarian rule for any $\varphi \in \Phi^R$, and a rule F is ***output-index-***

[5] For any $x, x' \in \mathbb{R}^m_+$, $y(x') \geq y(x)$ if $x' \geq x$ and $y(x') \geq y(x)$ if $x' \geq x$.

egalitarian if it is a φ-index egalitarian rule for any $\varphi \in \Phi^O$.

We say that a rule F is ***resource-egalitarian*** if for each $e = (\gamma, y, W) \in \mathcal{E}$, $F_1(e) = \cdots = F_n(e)$. Notice that, according to the definition, $S_i(e) = \frac{W}{n}$ for all $i \in N$. These rules award all agents the same amount of resources in each sector, but does not restrict each sector distribution. One extreme example of resource-egalitarian rule can be sector-t resource-egalitarianism: For some $t \in M$ and all $i \in N$, $F_{ti}(e) = \frac{W}{n}$ and $F_{si}(e) = 0$ for all $s \in M \backslash \{t\}$. Another extreme one would be resource-sector-egalitarianism: For all $i \in$ and $t \in M$, $F_{ti}(e) = \frac{W}{mn}$. One problem of resource-egalitarianism is that not all economies are applicable: the society does not have the full control on each agent's division rule. Thus, we need a refinement on the family of economies when we consider resource-egalitarianism. Define a class of economies:

$$\mathcal{E}^{RE} = \{(\gamma, y, W) \in \mathcal{E} : \gamma_i = \gamma_j \forall i, j \in N\}.$$

Due to the multi-commodity assumption, there are several ways to define output-egalitarianism. It can require full equality of all $y_{ti}(F_{ti}(e))$'s, or equality of $y_i(F_i(e))$'s. We take the second way, the loose requirement: a rule F is ***output-egalitarian*** if for all $e = (\gamma, y, W) \in \mathcal{E}$, $y_1(F_1(e)) = \cdots = y_n(F_n(e))$. This rule allocates resources in such a way that outputs are equalized across agents. Unfortunately, according to this definition, output-egalitarian rule is not valid for all the economies. For instance, consider an economy where agent i always distributes almost all her resource to the first sector with y_{1i} is relatively steeper than other y_{ti}'s, while agent j always distributes almost all her resouce to the second sector and y_{2j} is relatively steeper than other y_{tj}'s. Then it is impossible to equalize $y_i(F_i(e))$ and $y_j(F_j(e))$ since the allocation rule can control $S_i(e)$'s only. In other words, we need a refinement on the family of economies when we consider output-egalitarianism. Define a class of economies:

$$\mathcal{E}^{OE} = \{(\gamma, y, W) \in \mathcal{E} : \forall i, j \in N, S_i \in \mathbb{R}_+, \exists S_j \in \mathbb{R}_+ \; s.t. \; y_j(\gamma_j(S_j)) = y_i(\gamma_i(S_i))\}.$$

We introduce additional axioms for the sake of the axiomitization of the output-egalitarianism.

We call a transformation by a continuous and strictly increasing mapping $z_t : \mathbb{R}_+ \to \mathbb{R}_+$ with $z_t(0) = 0$. Let $\mathcal{Z}$ be the set of all transformations. We call $\mathcal{Z}^m$ the set of transformation profiles. For any transformation profile $z = (z_t)_{t \in M} \in \mathcal{Z}$ and any division function profile $\gamma = (\gamma_{ti})_{t \in M, i \in N} \in \Gamma^n$, we denote $z \circ \gamma_i = (z_t \circ \gamma_{ti})_{t \in M}$ a transformed division function for each agent i and $z \circ \gamma_S = (z \circ \gamma_i)_{i \in S}$ for each $S \subseteq N$. Similarly, for any transformation profile $z = (z_t)_{t \in M} \in \mathcal{Z}$ and any productivity function profile $y = (y_{ti})_{t \in M, i \in N} \in \mathcal{Y}^{mn}$, we denote $y_i \circ z^{-1} = (y_{ti} \circ z_t^{-1})_{t \in M}$ a transformed productivity function for each agent i and $y_S \circ z^{-1} = (y_i \circ z^{-1})_{i \in S}$ for each $S \subseteq N$. Notice that $z \circ \gamma \in \Gamma^n$ and $y \circ z^{-1} \in \mathcal{Y}^{mn}$.

Ordinality requires solutions to be invariant under any continuous and non-decreasing transformation of problems. It is a strengthening of scale invariance (formally defined in Moulin (2000), or Thomson (2003), for instance), which requires invariance under any linear transformation. Ordinality requires that if the total amount of resource is also transformed so that the original allocation (in terms of capacity) can be covered exactly, then we should stick to the original allocation. Therefore, *ordinality* is an invariance requirement which aims at cancelling the re-distributive effect of common 'unit' shocks unrelated to the underlying demands.

Ordinality. For all $(\gamma, y, W) \in \mathcal{E}$ and $z \in \mathcal{Z}^m$,

$$F(\gamma, y, W) = z(F(z \circ \gamma, y \circ z^{-1}, \sum_{t \in M, i \in N} z_t(F_{ti}(\gamma, y, W)))).$$

Resource-Ordinality requires that if everyone's productivity profile is transformed consistently, then the allocation should be the same as before.

Resource-Ordinality. For all $(\gamma, y, W) \in \mathcal{E}$ and $z \in \mathcal{Z}^m$,

$$F(\gamma, y, W) = F(\gamma, z \circ y, W).$$

Fixed point irrelevance says that a distribution $(F_i(e), y_i(F_i(e)))_{i \in N}$ as resource-output pairs of all the agents yielded by an allocation rule F from an economy e is achievable at a different economy e', then the allocation rule F should yield a same allocation in e'.

Fixed Point Irrelevance. For all $e = (\gamma, y, W), (\gamma', y', W) \in \mathcal{E}$ such that $\gamma'(S(e)) = \gamma(S(e))$ and $y'(\gamma'(S(e))) = y(\gamma(S(e)))$, $F(e') = F(e)$.

Proposition 3. *If a rule satisfies agreement and no-domination, then it satisfies fixed point irrelevance.*

Proof. Let any allocation rule F that satisfies *agreement* and *no-domination*. Then F is a φ-index egalitarian rule for some $\varphi \in \Phi$. Let any $e = (\gamma, y, W), e' = (\gamma', y', W) \in \mathcal{E}$ such that $\gamma'(S(e)) = \gamma(S(e))$ and $y'(\gamma'(S(e))) = y(\gamma(S(e)))$. We need to show that $F(e') = F(e)$. Let $\lambda = \varphi(F_i(e), y_i(F_i(e))) = \varphi(F_j(e), y_j(F_j(e)))$ be the equalized index value in economy e. Notice that $\varphi(F_i(e'), y_i'(F_i(e'))) = \varphi(F_j(e'), y_j'(F_j(e')))$ since F is φ-index egalitarian. Suppose conversely that

$F(e') \neq F(e)$. Note that $S(e') = S(e)$ implies $F(e') = F(e)$ from $\gamma'(S(e)) = \gamma(S(e))$. Therefore $S(e') \neq S(e)$, which implies that there exist $i, j \in N$ such that $S_i(e') > S_i(e)$ and $S_j(e') < S_j(e)$ from the fact that $\sum_{k\in N} S_k(e') = \sum_{k\in N} S_k(e)$. Then $F_i(e') = \gamma'(S_i(e')) > \gamma'(S_i(e)) = F_i(e)$ and $F_j(e') = \gamma'(S_j(e')) < \gamma'(S_j(e)) = F_j(e)$ by *sector monotonicity*. $y'_i(F_i(e)) = y_i(F_i(e)))$ implies $(F_i(e'), y'_i(F_i(e')) > (F_i(e), y_i(F_i(e)))$, which in turn implies $\varphi(F_i(e'), y'_i(F_i(e'))) > \lambda$, while $y'_j(F_i(e)) = y_j(F_j(e))$ implies $(F_i(e'), y'_i(F_i(e')) < (F_i(e), y_i(F_i(e))$, which in turn implies $\varphi(F_i(e'), y'_i(F_i(e'))) < \lambda$. That is, $\varphi(F_i(e'), y'_i(F_i(e'))) > \varphi(F_i(e'), y'_i(F_i(e')))$, which contradicts the fact that F is φ-index egalitarian.

□

Theorem 3. *A rule satisfies no-domination, agreement, and ordinality if and only if it is output-index-egalitarian.*

Proof. Let any *output-index-egalitarian* rule F. Let $\varphi \in \Phi^O$ be the index F equalizes. Then there exists $g \in \mathcal{G}$ such that $\varphi(a, b) = g(b)$ for each $a, b \in \mathbb{R}^m_+$. We first show that F satisfies all three axioms. Since it is *index-egalitarian*, it satisfies *no-domination* and *agreement* by Theorem **??**. Let any $z \in \mathcal{Z}^m$, $e = (\gamma, y, W) \in \mathcal{E}$, and $e^z = (z \circ \gamma, y \circ z^{-1}, \sum_{t\in M, i\in N} z_t(F_{ti}(e)))$. Let $x = F(e) \in \mathbb{R}^{mn}_+$. Since F equalizes the φ-value, for each $i, j \in N$, $\varphi(x_i, y_i(x_i)) = \varphi(x_j, y_j(x_j))$, that is, $g(y_i(x_i)) = g(y_j(x_j))$. From the fact that $\varphi(z(x_k), y \circ z^{-1}(z(x_k))) = g(y_k(x_k))$ for each $k \in N$, $z(x)$ equalizes the φ-value in e^z. Since the rule E^φ is uniquely defined, $F(e^z) = z(F(e))$, which indicates that F satisfies *ordinality*.

Let any F that satisfies *no-domination*, *agreement*, and *ordinality*. We show that F is *output-index-egalitarian*. By Theorem **??**, $F = E^\varphi$ for some $\varphi \in \Phi$. Let any $z \in \mathcal{Z}^m$, $e = (\gamma, y, W) \in \mathcal{E}$, and $e^z = (z \circ \gamma, y \circ$

$z^{-1}, \sum_{t\in M, i\in N} z_t(F_{ti}(e))))$. By *ordinality*, $F(e^z) = z(F(e))$, that is, for each $i \in N$, $\varphi(F(e), y(F(e))) = \varphi(z(F(e)), y(z^{-1}(z(F(e))))) = \varphi(z(F(e)), y(F(e)))$. Since $z \in \mathcal{Z}$ is arbitrary, $\varphi \in \Phi^O$, that is, F is *output-index-egalitarian.*[6] $\square$

Corollary 1. *Given a domain $\mathcal{E}^{OE}$, a rule satisfies no-domination, agreement, and ordinality if and only if it is output-egalitarian.*

Proof. It is trivial that any output-egalitarian rule satisfies *no-domination*, *agreement*, and *ordinality.* Let a rule F satisfy *no-domination*, *agreement*, and *ordinality.* We show that F is *output-egalitarian.* By Theorem **??**, $F = E^\varphi$ for some $\varphi \in \Phi^O$.

Let any $e \in \mathcal{E}^{OE}$ and any $i, j \in N$. To prove that F is *output-egalitarian*, it suffices to show that $y_i(F_i(e)) = y_j(F_j(e))$. Notice that since $\varphi \in \Phi^O$ and $\gamma_i, \gamma_j, y_i, y_j$ are strictly increasing, either $y_i(F_i(e)) = y_j(F_j(e))$, $y_i(F_i(e)) > y_j(F_j(e))$, or $y_i(F_i(e)) < y_j(F_j(e))$. Then, by the monotonicity property of φ, $\varphi(F_i(e), y_i(F_i(e))) = \varphi(F_j(e), y_j(F_j(e)))$ if and only if $y_i(F_i(e)) = y_j(F_j(e))$. $\square$

Theorem 4. *A rule satisfies no-domination, agreement, and resource-ordinality if and only if it is resource-index-egalitarian.*

Proof. Let any *resource-index-egalitarian* rule F. Let $\varphi \in \Phi^R$ be the index F equalizes. Then there exists $g \in \mathcal{G}$ such that $\varphi(a, b) = g(a)$ for each $a, b \in \mathbb{R}^m_+$. We first show that F satisfies all three axioms. Since it is *index-egalitarian*, it satisfies *no-domination* and *agreement* by Theorem **??**. Let

[6] To be specific, if, supposed conversely, $\varphi \in \Phi \backslash \{\Phi^O\}$, then there exist $a, a', b \in \mathbb{R}^m_+$ such that $a \neq a'$ and $\varphi(a, b) \neq \varphi(a', b)$. Then, for any $z \in \mathcal{Z}^m$ that $z(a) = a'$, any $e = (\gamma, y, W)$ that $(F_i(e), y_i(F_i(e))) = (a, b)$ for some $i \in N$, and $e^z = (z \circ \gamma, y \circ z^{-1}, \sum_{t\in M, i\in N} z_t(F_{ti}(e))))$, $\varphi(F_i(e), y_i(F_i(e))) = \varphi(a, b) \neq \varphi(a', b) = \varphi(z(F_i(e)), y_i(F_i(e)))$, which implies $F_i(e^z) \neq z(F_i(e))$, that is, F violates *ordinality.*

any $z \in \mathcal{Z}^m$, $e = (\gamma, y, W) \in \mathcal{E}$, and $e^z = (\gamma, z \circ y, W)$. Since F equalizes the φ-value, for each $i, j \in N$, $\varphi(F_i(e), y_i(F_i(e))) = \varphi(F_j(e), y_j(F_j(e)))$, that is, $g(F_i(e)) = g(F_j(e))$. From the fact that $\varphi(F_k(e^z), y_k(F_k(e^z))) = g(F_k(e^z))$ for each $k \in N$, $F(e)$ equalizes the φ-value in e^z. Since the rule E^φ is uniquely defined, $F(e^z) = F(e)$, which indicates that F satisfies *resource-ordinality*.

Let any F that satisfies *no-domination*, *agreement*, and *resource-ordinality*. We show that F is *resource-index-egalitarian*. By Theorem **??**, $F = E^\varphi$ for some $\varphi \in \Phi$. Let any $z \in \mathcal{Z}^m$, $e = (\gamma, y, W) \in \mathcal{E}$, and $e^z = (\gamma, z \circ y, W)$. By *resource-ordinality*, $F(e^z) = F(e)$, that is, for each $i \in N$, $\varphi(F(e), y(F(e))) = \varphi(F(e), z(y(F(e))))$. Since $z \in \mathcal{Z}$ is arbitrary, $\varphi \in \Phi^R$, that is, F is *resource-index-egalitarian*.

□

Corollary 2. *Given a domain $\mathcal{E}^{RE}$, a rule satisfies no-domination, agreement, and resource-ordinality if and only if it is resource-egalitarian.*

Proof. It is trivial that any resource-egalitarian rule satisfies *no-domination*, *agreement*, and *resource-ordinality*. Let a rule F satisfy *no-domination*, *agreement*, and *resource-ordinality*. We show that F is *resource-egalitarian*. By Theorem **??**, $F = E^\varphi$ for some $\varphi \in \Phi^R$.

Let any $e \in \mathcal{E}^{RE}$ and any $i, j \in N$. Then $\gamma_i = \gamma_j$. To prove that F is *resource-egalitarian*, it suffices to show that $F_i(e) = F_j(e)$. Since $\varphi \in \Phi^R$, $\varphi(F_i(e), y_i(F_i(e))) = \varphi(\gamma_i(S_i(e)), y_i(\gamma_i(S_i(e)))) = \varphi(\gamma_j(S_j(e)), y_j(\gamma_j(S_j(e)))) = \varphi(F_j(e), y_j(F_j(e)))$ if and only if $S_i(e) = S_j(e)$, that is, $F_i(e) = F_j(e)$.

□

4.3 Nash bargaining index - preliminary!

We define a refinement of the family of output-index. We call an output-index is *comprehensive-output-index* if it strictly increases in each sector output. That is, $\Phi^{CO} = \{\varphi \in \Phi^{O} : \exists g \in \mathcal{G} \ s.t. \ \forall a, b \in \mathbb{R}^m \cup \{(0, 0, \cdots, 0)\} \ \varphi(a, b) = g(b)$ and g_t strictly increasing in each $t \in M\}$. A rule F is ***comprehensive-output-index-egalitarian*** if it is a φ-index egalitarian rule for any $\varphi \in \Phi^{CO}$.

Sector Disability Monotonicity. For all $e = (\gamma, y, W) \in \mathcal{E}$, all $t \in M, i \in N$, and all $y'_{ti} \in \mathcal{Y}$, if $y'_{ti} \leq y_{ti}$ then $F_i(\gamma, (y'_i, y_{-i}), W) \geq F_i(\gamma, y, W)$, and if $y'_{ti} < y_{ti}$ then $F_i(\gamma, (y'_i, y_{-i}), W) > F_i(\gamma, y, W)$.

Corollary 3. *A rule satisfies no-domination, agreement, ordinality, and sector disability monotonicity if and only if it is comprehensive-output-index-egalitarian.*

Proof. Omit. □

One example of a noteworthy comprehensive-output-index is *Nash-bargaining-index*: $\varphi^{NB}(a, b) = \Pi_{t \in M} b_t$.

(Need to introduce a regarding axiom and a characterization result of *Nash-bargaining-index egalitarianism* here.)

5 Appendix

5.1 Proof of Theorem ??

Proof. Fix $\tilde{\gamma} = (\tilde{\gamma}_i)_{\{i \in N\}} \in \Gamma$ and $\tilde{y} = (\tilde{y}_i)_{\{i \in N\}} \in \mathcal{Y}^m$. Given a rule F and $\alpha \in \mathbb{R}_+$, let $\mathcal{E}(\alpha)$ be the set of economies where there exists an agent with $(\tilde{\gamma}, \tilde{y})$, that is, $\mathcal{E}(\alpha) = \{e \in \mathcal{E}:$ there exists $i \in N$ such that $(\gamma_i, y_i) = (\tilde{\gamma}, \tilde{y})$ and $S_i(e) = \alpha\}$. Notice that for any $e = (\gamma, y, W) \in \mathcal{E}(\alpha)$, any $j \in N$ with $(\gamma_j, y_j) = (\tilde{\gamma}, \tilde{y})$ receives α by *equal treatment of equals*, implied by *no-domination.*

Lemma 1. *If F satisfies no-domination and resource continuity, then for all $y \in \mathcal{Y}^{mn}$, all $N' \subset N$, and all $\alpha \in \mathbb{R}_+$, there exists $W^* \in \mathbb{R}_+$ such that $\sum_{i \in M} S_i(\gamma, y, W^*) = \alpha$.*

Proof. Let any $y \in \mathcal{Y}^{mn}$, $N' \subset N$, and $\alpha \geq 0$. Let $W_L \geq 0$ be such that $W_L < \alpha$. Since $\sum_{i \in N}(S_i(\gamma, y, W_L)) = W_L$ and for all $i \in N$ $S_i(\gamma, y, W_L) \geq 0$, $\sum_{i \in N'} S_i(\gamma, y, W_L) < \alpha$.

We next show that there is $W_H \geq 0$ such that $\sum_{i \in N'} S_i(\gamma, y, W_H) > \alpha$. Consider a sequence $(W^n \colon n \in \mathbb{N})$ such that $\lim_{n \to \infty} W^n = \infty$. Since $\sum_{i \in N} S_i(\gamma, y, W^n) = W^n$ for all n, there exists $j \in N$ such that $(S_j(\gamma, y, W^n) : n \in \mathbb{N})$ is an unbounded sequence, which implies, from *sector unboundedness*, that $(F_{tj}(\gamma, y, W^n) : n \in \mathbb{N})$ is unbounded for each $t \in M$. Then, for each $t \in M$, $(y_{tj}(F_{1j}(\gamma, y, W^n)) : n \in \mathbb{N})$ is also an unbounded sequence since $y_{tj}(\cdot)$ is an unbounded function.

We show that there is $\bar{n}$ such that $\sum_{i \in N'} S_i(\gamma, y, W^{\bar{n}}) > \alpha$. Suppose by contradiction that $\sum_{i \in N'} S_i(\gamma, y, W^n) \leq \alpha$ for all $n \in \mathbb{N}$. Since $(F_{tj}(\gamma, y, W^n) : n \in \mathbb{N})$ and $(y_{tj}(F_{tj}(\gamma, y, W^n)) : n \in \mathbb{N})$ are unbounded for each t, there exists n such that $\alpha < F_{tj}(\gamma, y, W^n)$ for all t, and $(y_{1i}(\alpha), \cdots, y_{mi}(\alpha)) < (y_{1j}(F_{1j}(\gamma, y, W^n)), \cdots, y_{mj}(F_{mj}(\gamma, y, W^n)))$ for all $i \in N'$. Hence for such n, for all $t \in M, i \in N'$, $F_{ti}(\gamma, y, W^n) \le \alpha < F_{tj}(\gamma, y, W^n)$ and $y_{ti}(F_{ti}(\gamma, y, W^n)) \le y_{ti}(\alpha) < y_{tj}(F_{tj}(\gamma, y, W^n))$, which contradicts *no-domination.*

Now let $W_H \equiv W^{\bar{n}}$. Then $\sum_{i \in N'} S_i(\gamma, y, W_H) > \alpha$. Since $\sum_{i \in N'} S_i(\gamma, y, W_L) < \alpha < \sum_{i \in N'} S_i(\gamma, y, W_H)$, by *resource continuity,* there exists $W^* \in \mathbb{R}_+$ such that $\sum_{i \in N'} S_i(\gamma, y, W^*) = \alpha$.

□

Lemma 2. *Assume that F satisfies no-domination and agreement. For all $e \equiv (\gamma, y, W)$ and all three distinct $i, j, k \in N$, there is $e' \equiv (\gamma', y', W')$ such that $(\gamma'_i, y'_i) = (\gamma_i, y_i)$, $(\gamma'_j, y'_j) = (\gamma'_k, y'_k) = (\gamma_j, y_j)$, $F_i(e') = F_i(e)$, and $F_j(e') = F_k(e') = F_j(e)$.*

Proof. Let $e = (\gamma, y, W)$ and i, j, k are distinct. Let y' be such that $y'_i = y_i$, $y'_j = y'_k = y_j$. By Lemma **??**, there is W' such that $S_i(e') + S_j(e') = S_i(e) + S_j(e)$, where $e' \equiv (\gamma', y', W')$. By *separability* (implied by *agreement*), $F_i(e') = F_i(e)$ and $F_j(e') = F_j(e)$. Since $(\gamma'_j, y'_j) = (\gamma'_k, y'_k)$ and by *equal treatment of equals* (implied by *no-domination*), $F_k(e') = F_j(e') = F_j(e)$.

□

Lemma 3. *If F satisfies no-domination and agreement, then for any $\alpha \in \mathbb{R}_+$ and any $e, e' \in \mathcal{E}(\alpha)$, there is no pair $i, j \in N$ such that $(F_i(e), y_i(F_i(e))) \le (F_j(e'), y_j(F_j(e')))$ and $(F_{ti}(e), y_{ti}(F_{ti}(e))) < (F_{tj}(e'), y_{tj}(F_{tj}(e')))$ for some $t \in M$.*

Proof. Suppose conversely that, without loss of generality, there exist $\alpha \ge 0, i, j \in N$, and $e = (\gamma, y, W), e' = (\gamma', y', W') \in \mathcal{E}(\alpha)$ such that $(F_i(e), y_i(F_i(e))) \le$

$(F_j(e'), y_j(F_j(e')))$ and $(F_{1i}(e), y_{1i}(F_{1i}(e))) < (F_{1j}(e'), y_{1j}(F_{1j}(e')))$. By Lemma **??**, we may let $(\gamma_1, y_1) = (\gamma_1', y_1') = (\tilde{\gamma}, \tilde{y})$ and assume that $1, i, j$ are three distinct agents. Note that $S_1(e) = S_1(e') = \alpha$ and $F_1(e) = F_1(e') = \tilde{\gamma}(\alpha)$. Let $\hat{\gamma}$ and $\hat{y}$ be such that $(\hat{\gamma}_{\{1,i,j\}}, \hat{y}_{\{1,i,j\}}) = (\gamma'_{\{1,i,j\}}, y'_{\{1,i,j\}})$ and $(\hat{\gamma}_{N\setminus\{1,i,j\}}, \hat{y}_{N\setminus\{1,i,j\}}) = (\gamma_{N\setminus\{1,i,j\}}, y_{N\setminus\{1,i,j\}})$. By Lemma **??**, there is $\hat{W}$ such that $\hat{e} \equiv (\hat{\gamma}, \hat{y}, \hat{W})$ and $S_1(\hat{e}) + S_i(\hat{e}) + S_j(\hat{e}) = S_1(e') + S_i(e') + S_j(e')$. By *separability* (implied by *agreement*), $F_{\{1,i,j\}}(\hat{e}) = F_{\{1,i,j\}}(e')$.

Let γ'' and y'' such that $(\gamma_i'', y_i'') = (\gamma_i, y_i)$, $(\gamma_j'', y_j'') = (\gamma_j', y_j')$, $(\gamma_1'', y_1'') = (\tilde{\gamma}, \tilde{y})$, and for all $h \neq 1, i, j$, $(\gamma_h'', y_h'') = (\gamma_h, y_h)$. By Lemma **??**, there is $W'' \geq 0$ such that $e'' \equiv (\gamma'', y'', W'')$ and

$$S_1(e'') + S_i(e'') + S_j(e'') = \alpha + S_i(e) + S_j(e'). \tag{1}$$

Suppose $S_1(e'') > \alpha$. By applying *agreement* to e and e'', we get $S_i(e'') > S_i(e)$. Likewise, by applying *agreement* to $\hat{e}$ and e'', we get $S_j(e'') > S_j(e')$. Altogether, $S_1(e'') + S_i(e'') + S_j(e'') > \alpha + S_i(e) + S_j(e')$, contradicting (**??**). Therefore $S_1(e'') \leq \alpha$. Similarly, we can show $S_1(e'') \geq \alpha$. Hence $S_1(e'') = \alpha$.

Then, by applying *separability* to e'', $F_i(e'') = F_i(e)$ and $F_j(e'') = F_j(e')$. That is, $(F_i(e''), y_i''(F_i(e''))) = (F_i(e), y_i(F_i(e)))$, $(F_j(e''), y_j''(F_j(e''))) = (F_j(e'), y_j'(F_j(e')))$. That, however, implies $(F_i(e''), y_i(F_i(e''))) \leq (F_j(e''), y_j(F_j(e'')))$ and $(F_{1i}(e''), y_{1i}(F_{1i}(e''))) < (F_{1j}(e''), y_{1j}(F_{1j}(e'')))$, which violates *no-domination* at e''. $\square$

Let $C(\alpha)$ be the set of all resource-output pairs for all sectors in all economies in $\mathcal{E}(\alpha)$, that is, $C(\alpha) = \{c \in \mathbb{R}^{2m}_+ :$ there exists $e \in \mathcal{E}(\alpha)$ such that $c = (F_i(e), y_i(F_i(e)))$ for some $i \in N\}$.

Also, for each $t \in M$, let

$\Gamma\mathcal{Y}_t : \{((\gamma_t'(\cdot), \tilde{\gamma}_{-t}), (y_t', \tilde{y}_{-t})) : \mathbb{R}_+ \to \Gamma \times \mathcal{Y}^m : \exists (\gamma, y) \in \Gamma^n \times \mathcal{Y}^{mn}$ and $i, j \in$

N such that $(\gamma_i, y_i) = (\tilde{\gamma}, \tilde{y})$ and $F_{ti}(((\gamma'_t(W), \tilde{\gamma}_{-t}), \gamma_{-j}), ((y'_t, \tilde{y}_{-t}), y_{-j}), W) = F_{tj}(((\gamma'_t(W), \tilde{\gamma}_{-t}), \gamma_{-j}), ((y'_t, \tilde{y}_{-t}), y_{-j}), W)$ for each $W \in \mathbb{R}_+\}$.

Moreover, for each $t \in M$, let $C_t(\alpha)$ be the set of all sector-t resource-output pairs of all agents $j \in N$ with (γ_j, y_j) corresponds to $\Gamma\mathcal{Y}_t$ in all economies in $\mathcal{E}(\alpha)$, that is,
$C_t(\alpha) = \{(a,b) \in \mathbb{R}^2_+ : \exists e = (\gamma, y, W) \in \mathcal{E}(\alpha)$ and $j \in N$ such that $(\gamma_j, y_j) = ((\gamma'_t(W), \tilde{\gamma}_{-t}), (y'_t, \tilde{y}_{-t}))$ for some $((\gamma'_t(W), \tilde{\gamma}_{-t}), (y'_t, \tilde{y}_{-t})) \in \Gamma\mathcal{Y}_t$ and $(F_{tj}(e), y_{tj}(F_{tj}(e))) = (a,b)\}$.

We show that for all $\alpha \geq 0$ and each $t \in M$, $C_t(\alpha)$ is downward sloping.

Lemma 4. *If F satisfies no-domination and agreement, then for any $\alpha \geq 0$ and $t \in M$, $C_t(\alpha)$ is downward sloping, that is, for all $(a,b), (a',b') \in C_t(\alpha)$ with $a' > a$, we have $b' \leq b$.*

Proof. Assume that F satisfies *no-domination* and *agreement*. To prove that $C_t(\alpha)$ is downward sloping, suppose to the contrary that for some $(a,b), (a',b') \in C_t(\alpha)$, $(a',b') > (a,b)$. By definition of $C_t(\alpha)$, there exist $e = (\gamma, y, W), e' = (\gamma', y', W') \in \mathcal{E}(\alpha)$ such that for some $i, j \in N$, $(F_i(e), y_i(F_i(e))) = (a, \tilde{\gamma}_{-t}(\alpha), b, \tilde{y}_{-t}(\tilde{\gamma}_{-t}(\alpha)))$ and $(F_j(e'), y_j(F_j(e'))) = (a', \tilde{\gamma}_{-t}(\alpha), b', \tilde{y}_{-t}(\tilde{\gamma}_{-t}(\alpha)))$. That is, $(F_{ti}(e), y_{ti}(F_{ti}(e))) < (F_{tj}(e'), y_{tj}(F_{1j}(e')))$ and $(F_{si}(e), y_{si}(F_{si}(e))) = (F_{sj}(e'), y_{sj}(F_{sj}(e')))$ for all $s \in M \backslash \{t\}$, which contradicts Lemma **??**. $\square$

The next lemma, which says that $C_t(\alpha)$ and $C_t(\alpha')$ for each $t \in M$ and $\alpha \neq \alpha'$ does not intersect, can be established from the above lemmas as in Chun, Jang, and Ju (2014).

Lemma 5. *For each $t \in M$, $\{C_t(\alpha) : \alpha \in \mathbb{R}_+\}$ is a collection of disjoint sets.*

The next lemma says that, for each $t \in M$, by varying $\alpha \in \mathbb{R}_+$, $C_t(\alpha)$'s can cover the positive quadrant.

Lemma 6. *For all* $(a, b) \in \mathbb{R}^2_{++} \cup \{(0,0)\}$ *and for each* $t \in M$*, there is a unique* $\alpha \geq 0$ *such that* $(a, b) \in C_t(\alpha)$.

Proof. Fix any $s \in M$. Also let any $(a, b) \in \mathbb{R}^2_{++} \cup \{(0,0)\}$. Let B be such that $b = B \cdot \tilde{y}_s(\tilde{\gamma}_s(a))$. Without loss of generality, consider that $B \geq 1$.[7] Let any $(\gamma, y) \in \Gamma^n \times \mathcal{Y}^{mn}$ such that $(\gamma_i, y_i) = (\tilde{\gamma}, \tilde{y})$ and $(\gamma_j, y_j) = (\tilde{\gamma}, (B\tilde{y}_s, \tilde{y}_{-s}))$ for some $i, j \in N$. Notice that $y_{sj}(a) = b$. Also, for each $A > 0$, let $e^A \equiv (\gamma^A, y, W) \equiv ((A \cdot \tilde{\gamma}_s, \tilde{\gamma}_{-s}), \gamma_{-j}), y, W) \in \mathcal{E}$. We first show the following Claim.

Claim 1. *If* $A > 1$ *then* $F_{tj}(e^A) \leq F_{ti}(e^A)$ *for any* $t \in M \backslash \{s\}$.

Proof. Consider any $A > 1$ and suppose conversely that there exists some $r \in M \backslash \{s\}$ such that $F_{rj}(e^A) > F_{ri}(e^A)$. Then $F_{tj}(e^A) > F_{ti}(e^A)$ for all $t \in M \backslash \{s\}$ since $\gamma^A_{tj} = \gamma^A_{ti}$ for all $t \in M \backslash \{s\}$. Moreover, since $A > 1$, $\dfrac{F_{sj}(e^A)}{F_{rj}(e^A)} > \dfrac{F_{si}(e^A)}{F_{ri}(e^A)}$, which implies $F_{sj}(e^A) > F_{si}(e^A)$. That is, $F_j(e^A) > F_i(e^A)$. Since $y_j = (B\tilde{y}_s, \tilde{y}_{-s}) \geq \tilde{y} = y_i$, $(F_j(e^A), y_j(F_j(e^A))) > (F_i(e^A), y_i(F_i(e^A)))$, which indicates that *no domination* is violated. □

Let $w^* \in \mathbb{R}_+$ be such that $B\tilde{y}_s(w^*) = y_{si}(F_{si}(e^A))$, w be such that $w - w^* = \sum_{t \in M \backslash \{s\}} F_{ti}(e^A)$, and let $C = \sup\{c \in \mathbb{R} : c\tilde{\gamma}_s(w) = w^*\} > 0$. That is, if j's division function is $(C \cdot \tilde{\gamma}_s, \tilde{\gamma}_{-s})$, then it divides an individual resource w into $(w_1, \cdots, w_m) = (w^*, \tilde{\gamma}_{-s}(w))$ so that $(y_{1j}(w_1), \cdots, y_{2j}(w_m)) = (B\tilde{y}_s(w^*), \tilde{y}_{-s}(\tilde{\gamma}_{-s}(w))) = (y_{1i}(F_{1i}(e^A)), \cdots, y_{mi}(F_{mi}(e^A)))$. Notice that $C \leq$

[7] The proof considering $B < 1$ is analogous.

1 since $B\tilde{y}_s \geq y_{si}$ and $B\tilde{y}_s(w^*) = y_{si}(F_{si}(e^A))$ so that $w^* \leq F_{si}(e^A) = \tilde{\gamma}_s(w)$. We show the following Claim.

Claim 2. *If $A \in (0, C)$, then $F_{tj}(e^A) \geq F_{ti}(e^A)$ for any $t \in M \backslash \{s\}$.*

Proof. Let any $A \in (0, C)$ and suppose conversely that there exists some $r \in M\backslash\{s\}$ such that $F_{rj}(e^A) < F_{ri}(e^A)$. Then $\gamma^A_{tj} = \gamma^A_{ti}$ for all $t \in M\backslash\{s\}$ implies $F_{tj}(e^A) < F_{ti}(e^A)$ for all $t \in M\backslash\{s\}$, and, in turn, $y_{tj} = y_{ti}$ implies $y_{tj}(F_{tj}(e^A)) < y_{ti}(F_{ti}(e^A))$ for all $t \in M\backslash\{s\}$. Moreover, $F_{sj}(e^A) < F_{si}(e^A)$ since $A < 1$, and $y_{sj}(F_{sj}(e^A)) \leq B\tilde{y}(A\tilde{\gamma}_s(w^*)) < y_{si}(F_{si}(e^A))$ by the construction of C and from $A < C$. Therefore $F_j(e^A) < F_i(e^A)$ and $y_j(F_j(e^A)) < y_i(F_i(e^A))$. That is, $(F_j(e^A), y_j(F_j(e^A))) < (F_i(e^A), y_i(F_i(e^A)))$, which contradicts *no domination*. □

From Claim 1 and 2 and by division function invariance, for each $W \in \mathbb{R}_+$, there exists $A^*(W)$ such that $F_{tj}(((A^*(W)\cdot\tilde{\gamma}_s, \tilde{\gamma}_{-s}), \gamma_{-j}), y, W) = F_{ti}(((A^*(W)\cdot \tilde{\gamma}_s, \tilde{\gamma}_{-s}), \gamma_{-j}), y, W)$ for all $t \in M\backslash\{s\}$. Finally, as we denote $\gamma^*(\cdot) \equiv (A^*(\cdot)\tilde{\gamma}_s, \tilde{\gamma}_{-s})$, $\gamma^*(W) \in \Gamma$ for each $W \in \mathbb{R}_+$ is implied by F's *sector unboundedness* and *resource monotonicity*. Thus $((\gamma^*(\cdot), y_j) \in \Gamma\mathcal{Y}_s$. Let $e^* = ((\gamma^*(W), \gamma_{-j}), y, W)$. By Lemma **??** and from $y_{sj}(a) = b$, there exists $W \in \mathbb{R}_+$ such that $(F_{sj}(e^*), y_{sj}(F_{sj}(e^*) = (a, b)$, which implies that $(a, b) \in C_s(\alpha)$ by letting $\alpha = S_i(e^*)$.

Finally, the uniqueness of α is implied by Lemma **??**. □

The next lemma says that for each $t \in M$, if $\alpha_1 > \alpha_2$, then $C_t(\alpha_1)$ lies above $C_t(\alpha_2)$, which can be shown using Lemma **??**, **??**, and **??** as in Chun, Jang, and Ju (2014).

Lemma 7. *For each $t \in M$, if $\alpha_1 > \alpha_2$, then* *(i) for all $(a,b) \in C_t(\alpha_2)$ there exists $(a',b') \in C_t(\alpha_1)$ such that $(a,b) < (a',b')$, and*

(ii) there is no $(a'',b'') \in C_t(\alpha_2)$ and $(a,b) \in C_t(\alpha_1)$ such that $(a'',b'') > (a,b)$.

Now we define the following set:
$D_2(\alpha) = \{(\alpha_1,\alpha_2) \in \mathbb{R}^2_+ : \exists e = (\gamma,y,W) \in \mathcal{E}(\alpha)$ and $j \in N$ such that $(F_{tj}(e), y_{tj}(F_{tj}(e))) \in C_t(\alpha_t)$ for each $t = 1,2$ and $(\gamma_{t'j}, y_{t'j}) = (\gamma_{tj}, y_{tj})$ for all $t' > t\}$.

We first show that for each $\alpha \in \mathbb{R}_+$, $D_2(\alpha)$ is downward sloping.

Lemma 8. *For any $\alpha \in \mathbb{R}_+$, $D_2(\alpha)$ is downward sloping.*

Proof. To prove that $D_2(\alpha)$ is downward sloping, suppose, to the contrary, that for some $(\alpha_1,\alpha_2), (\alpha'_1,\alpha'_2) \in D_2(\alpha)$, $(\alpha_1,\alpha_2) < (\alpha'_1,\alpha'_2)$. By Lemma **??**, there exist $(a_k,b_k) \in C_k(\alpha_k)$ and $(a'_k,b'_k) \in C_k(\alpha'_k)$ such that $(a_k,b_k) < (a'_k,b'_k)$ for each $k \in \{1,2\}$. By definition of $D_2(\alpha)$, there exist $e, e' \in \mathcal{E}(\alpha)$ and $i,j \in N$ such that $(F_i(e), y_i(F_i(e))) = (a_1, a_2, \cdots, a_2, b_1, b_2, \cdots, b_2)$ and $(F_j(e'), y_j(F_j(e'))) = (a'_1, a'_2, \cdots, a'_2, b'_1, b'_2, \cdots, b'_2)$. That is, $(F_{ti}(e), y_{ti}(F_{ti}(e))) < (F_{tj}(e'), y_{tj}(F_{tj}(e')))$ for all $t \in M$, which contradicts Lemma **??**. □

Lemma 9. *$\{D_2(\alpha) : \alpha \in \mathbb{R}_+\}$ is a collection of disjoint sets.*

Proof. Let $\alpha' > \alpha$ and suppose that $(\alpha_1,\alpha_2) \in D_2(\alpha) \cap D_2(\alpha')$. Then there exists $e = (\gamma,y,W) \in \mathcal{E}(\alpha)$ and $i \in N\backslash\{1\}$ such that $(\gamma_1,y_1) = (\tilde{\gamma},\tilde{y})$, $S_1(e) = \alpha$, $(F_{1i}(e), y_{1i}(F_{1i}(e))) \in C_1(\alpha_1)$, and $(F_{ti}(e), y_{ti}(F_{ti}(e))) \in C_2(\alpha_2)$ for all $t \in M\backslash\{1\}$. By Lemma **??**, there is W' such that $e' = (\gamma, y, W')$ and $S_1(e') = \alpha'$. Then $W' > W$ by *resource monotonicity*, and therefore $F_i(e') > F_i(e)$, which also implies $y_i(F_i(e')) > y_i(F_i(e))$. Since $(F_{1i}(e'), y_{1i}(F_{1i}(e'))) >$

$(F_{1i}(e), y_{1i}(F_{1i}(e)))$, $(F_{2i}(e'), y_{2i}(F_{2i}(e'))) > (F_{2i}(e), y_{2i}(F_{2i}(e)))$, and by Lemma **??**, there exist $\alpha'_1 > \alpha_1$ and $\alpha'_2 > \alpha_2$ such that $(F_{1i}(e'), y_{1i}(F_{1i}(e'))) \in C_1(\alpha'_1)$ and $(F_{2i}(e'), y_{2i}(F_{2i}(e'))) \in C_2(\alpha'_2)$. Since $(\alpha_1, \alpha_2) \in D_2(\alpha')$ and $(\alpha'_1, \alpha'_2) \in D_2(\alpha')$, $D_2(\alpha')$ is not downward sloping, contradicting Lemma **??**. □

The next lemma says that $D_2(\alpha)$ can cover the positive quadrant by varying $\alpha \geq 0$.

Lemma 10. *For all $(\alpha_1, \alpha_2) \in \mathbb{R}^2_+$, there is a unique $\alpha \in \mathbb{R}_+$ such that $(\alpha_1, \alpha_2) \in D_2(\alpha)$.*

Proof. Let any $(\alpha_1, \alpha_2) \in \mathbb{R}^2_+, (a_1, b_1) \in C_1(\alpha_1)$, and $(a_2, b_2) \in C_2(\alpha_2)$. Let any $(\gamma, y) \in \Gamma^n \times \mathcal{Y}^{mn}$ such that $(\gamma_1, y_1) = (\tilde{\gamma}, \tilde{y}), (\gamma_{2i}, y_{2i}) = \cdots = (\gamma_{mi}, y_{mi})$ with $\gamma_i(a_1 + (m-1) \cdot a_2) = (a_1, a_2, \cdots, a_2)$, $y_{1i}(a_1) = b_1$, and $y_{2i}(a_2) = b_2$ for some $i \in N \backslash \{1\}$. By Lemma **??**, there exists W such that $e = (\gamma, y, W)$ and $S_i(e) = a_1 + (n-1) \cdot a_2$, which implies that $(F_i(e), y_i(F_i(e))) = (a_1, a_2, \cdots, a_2, b_1, b_2, \cdots, b_2)$. By letting $\alpha = S_1(e)$ and from $(a_1, b_1) \in C_1(\alpha_1)$ and $(a_2, b_2) \in C_2(\alpha_2)$, we have $(\alpha_1, \alpha_2) \in D_2(\alpha)$. The uniqueness of α is implied by Lemma **??**. □

The next lemma says that $D_2(\alpha_1)$ lies above $D_2(\alpha_2)$ for any $\alpha_1 > \alpha_2$.

Lemma 11. *If $\alpha' > \alpha$, then for all $(\alpha_1, \alpha_2) \in D_2(\alpha)$,*

(i) there exists $(\alpha'_1, \alpha'_2) \in D_2(\alpha')$ such that $(\alpha_1, \alpha_2) < (\alpha'_1, \alpha'_2)$, and

(ii) there is no $(\alpha''_1, \alpha''_2) \in D_2(\alpha')$ such that $(\alpha''_1, \alpha''_2) < (\alpha_1, \alpha_2)$.

Proof. Let any $\alpha' > \alpha$ and any $(\alpha_1, \alpha_2) \in D_2(\alpha)$. We first show *(i)*. There exists $e = (\gamma, y, W) \in \mathcal{E}(\alpha)$ such that $(\gamma_1, y_1) = (\tilde{\gamma}, \tilde{y})$, $S_1(e) = \alpha$, and $(F_{ti}(e), y_{ti}(F_{ti}(e))) \in C_t(\alpha_t)$ for $t = 1, 2$. By Lemma **??**, there exists W' such that $e' = (\gamma, y, W')$ and $F_1(e') = \alpha'$. By *resource monotonicity*, $(F_{ti}(e'), y_{ti}(F_{ti}(e'))) > (F_{ti}(e), y_{ti}(F_{ti}(e)))$ for $t = 1, 2$. By Lemma **??**, for each $t = 1, 2$, there exists a unique $\alpha'_t > \alpha_t$ such that $(F_{ti}(e'), y_{ti}(F_{ti}(e'))) \in C_t(\alpha'_t)$. Therefore, $(\alpha'_1, \alpha'_2) \in D_2(\alpha')$.

To show (ii), suppose conversely that there exists $(\alpha''_1, \alpha''_2) \in D_2(\alpha')$ such that $(\alpha''_1, \alpha''_2) < (\alpha_1, \alpha_2)$. By *(i)* and from $\alpha' > \alpha$, there exists $(\beta_1, \beta_2) \in D_2(\alpha)$ such that $(\alpha''_1, \alpha''_2) > (\beta_1, \beta_2)$. That is, $(\alpha_1, \alpha_2) > (\alpha''_1, \alpha''_2) > (\beta_1, \beta_2)$ and $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in D_2(\alpha)$, which contradicts Lemma **??**. □

Now we define $D_t(\cdot)$ for $t > 2$. For each $t \in M \backslash \{1, 2\}$,
$D_t(\alpha) = \{(\alpha_{t-1}, \alpha_t) \in \mathbb{R}^2_+ : \exists e = (\gamma, y, W) \in \mathcal{E}(\alpha), j \in N$ with $(F_{tj}(e), y_{tj}(F_{tj}(e))) \in C_t(\alpha_t)$ and $(\gamma_{t'j}, y_{t'j}) = (\gamma_{tj}, y_{tj})$ for all $t' > t$ such that, $(F_{t'j}(e), y_{t'j}(F_{t'j}(e))) \in C_{t'}(\alpha'_{t'})$ for $t' < t$ and $(\alpha'_1, \alpha'_2) \in D_2(\alpha_2), \cdots, (\alpha_{t-2}, \alpha'_{t-1}) \in D(\alpha_{t-1})\}$. Notice that for such $j \in N$ in the definition of $D_t(\alpha)$, $(F_{tj}(e), y_{tj}(F_{tj}(e))) = (F_{t'j}(e), y_{tj}(F_{t'j}(e)))$ for all $t' > t$.

Following the process of Lemma **??**, **??**, **??**, and **??**, we can show the following lemma.

Lemma 12. *For any $t \in \{2, \cdots, m\}$,*

(i) $D_t(\alpha)$ is downward sloping for each $\alpha \in \mathbb{R}_+$.

(ii) $\{D_t(\alpha) : \alpha \in \mathbb{R}_+\}$ is a collection of disjoint sets.

(iii) For all $(\alpha_1, \alpha_2) \in \mathbb{R}^2_+$, there is a unique $\alpha \geq 0$ such that $(\alpha_1, \alpha_2) \in D_t(\alpha)$.

(iv) If $\alpha' > \alpha$, then for all $(\alpha_1, \alpha_2) \in D_t(\alpha)$,

(iv-i) there exists $(\alpha'_1, \alpha'_2) \in D_t(\alpha')$ such that $(\alpha_1, \alpha_2) < (\alpha'_1, \alpha'_2)$, and

(iv-ii) there is no $(\alpha''_1, \alpha''_2) \in D_t(\alpha')$ *such that* $(\alpha''_1, \alpha''_2) < (\alpha_1, \alpha_2)$.

Now we are ready to prove Theorem 1.

For each $t \in \{2, \cdots, m\}$, define $\delta_t : \mathbb{R}^2_+ \to \mathbb{R}_+$ by $\delta_t(\alpha', \alpha'') = \alpha$, where $\alpha \in \mathbb{R}_+$ is the unique number for which $(\alpha', \alpha'') \in D_t(\alpha)$, and define $\delta : \mathbb{R}^m_+ \to \mathbb{R}_+$ by $\delta(\alpha_1, \cdots, \alpha_m) = \delta_m(\delta_{m-1}(\cdots \delta_3(\delta_2(\alpha_1, \alpha_2), \alpha_3), \cdots, \alpha_m)$. Notice that for each $(\alpha_1, \cdots, \alpha_m) \in \mathbb{R}^m_+$, $\delta(\alpha_1, \cdots, \alpha_m)$ is uniquely determined, $\delta(0, \cdots, 0) = 0$, and $\delta(\alpha_1, \cdots, \alpha_m) > 0$ if and only if $(\alpha_1, \cdots, \alpha_m) \geq 0$ with $\alpha_t > 0$ for some $t \in M$. From the preceding discussion, we can conclude that δ is continuous, nondecreasing, and for any $(\alpha_1, \cdots, \alpha_m) < (\alpha'_1, \cdots, \alpha'_m), \delta(\alpha_1, \cdots, \alpha_m) < \delta(\alpha'_1, \cdots, \alpha'_m)$.

Define $\varphi : \mathbb{R}^{2m}_{++} \cup \{(0, 0, 0, 0)\} \to \mathbb{R}_+$ by $\varphi(a_1, \cdots, a_m, b_1, \cdots, b_m) = \delta(\alpha_1, \cdots, \alpha_m)$, where α_t is the unique number for which $(a_t, b_t) \in D_t(\alpha_t)$ for each $t \in M$. Then it is obvious that $\varphi \in \Phi$, according to the property of δ.

We now show that $F(e) = E^\varphi(e)$ for all $e = (\gamma, y, W) \in \mathcal{E}$. If $(\gamma_i, y_i) = (\tilde{\gamma}, \tilde{y})$ for some $i \in N$, then by letting $\lambda = S_i(e)$, for all $j \in N$, we have $(\alpha_{tj})_{\{t \in M\}}$ such that $(F_{tj}(e), y_{tj}(F_{tj}(e))) \in C_t(\alpha_{tj})$ for each $t \in M$, and $\varphi(F_j(e), y_j(F_j(e))) = \delta(\alpha_{1j}, \cdots, \alpha_{mj}) = \lambda$. Since $\sum_{j \in N} S_j(e) = W$, $F(e) = E^\varphi(e)$.

We now consider the case that there is no $i \in N$ with $(\gamma_i, y_i) = (\tilde{\gamma}, \tilde{y})$. Consider $\gamma' = (\tilde{\gamma}, \gamma_2, \cdots, \gamma_n)$ and $y' = (\tilde{y}, y_2, \cdots, y_n)$. By Lemma 1, there exists W' such that $e' = (\gamma', y', W')$ and $\sum_{i \in N \setminus \{1\}} S_i(e') = \sum_{i \in N \setminus \{1\}} S_i(e)$. By *separability* (implied by *agreement*), for all $i \in N \setminus \{1\}$, $S_i(e') = S_i(e)$, and therefore $F_i(e') = F_i(e)$. Let $\lambda = S_1(e')$. Then, for all $i \in N \setminus \{1\}$, $\varphi(F_i(e), y_i(F_i(e))) = \varphi(F_i(e'), y_i(F_i(e'))) = \lambda$. Similarly, we can show that

$\varphi(F_1(e), y_1(F_1(e))) = \lambda$. Therefore, $F(e) = E^{\varphi}(e)$.

$\square$