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Valuing Income-Contingent Loans as Path-Dependent Options

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An income-contingent loan (ICL) contract can be viewed as a series of path-dependent options, and its value can be determined by applying an option pricing technique. This paper discusses conceptual and technical issues related to this approach and using the approach, determines the value of ICL contracts offered to university students in Korea in 2010. The value of an ICL contract is contrasted to the value of a comparable standard loan (SL) contract. Also, the option-based valuation is compared to the utility-based valuation.

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Keywords: Income-Contingent Loan, Path-Dependent Option, Certainty Equivalent

I. Introduction

An income-contingent loan (ICL) was first conceived by Friedman and Kuznets (1945) as a possible way to finance the training of professionals such as lawyers and doctors. Given high financial reward for lawyers and doctors, there is no reason for the government to subsidize the training of these professionals. To put it another way, the externality associated with the education of these professionals is not significant; thus there is no necessity for government intervention. However, if financial markets are not functioning well, perhaps due to asymmetric information, qualified individuals may not be able to get adequate financing from financial markets. In this case, the government may be justified to provide financing in some form. Friedman and Kuznets argued that equity-financing is more appropriate than debt-financing for the professional training. The idea is that the government owns the "shares" of these individuals, just as shareholders own the shares of corporations.

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When these individuals earn income, the government takes a fraction of the income. When they do not earn income, the government does not take any money back. It took many decades for the idea of ICLs to become a reality. It was Friedman (1955) who proposed that ICLs be offered to university students. In the 1970s, Yale University offered what is called 'Tuition Postpone Option' which has all the important features of ICLs (Nerlove, 1975). Since the late 1980s, nation-wide ICL programs have been introduced in Australia, New Zealand, UK, South Africa, and Chile (Chapman, 1997, 2006). In Korea, standard government-sponsored loans (SLs) for university students began to be replaced with ICLs in 2010.[1]

As nation-wide ICL programs got implemented, the implication on the national budget became a serious concern. In the original proposal of Friedman and Kuznets (1945) and Friedman (1955), ICLs were to be offered without government subsidy. If there is no government subsidy, then there would be no serious implication on the national budget. In reality, ICLs are offered with significant government subsidy, and the budgetary implication cannot be ignored. An assessment of the true cost of an ICL program is not an easy task, however. An ICL contract is a combination of a loan contract and an insurance contract. Neither the value of a loan contract nor that of an insurance contract can be easily determined. Migali (2010) calculated the value of an ICL within the expected utility framework. Assuming a standard expected utility function with a range of parameter values, he computed the expected utility from an ICL. From the expected utility, one can derive a certainty-equivalent measure of the value of the ICL, from which one can infer the "market price" of the ICL.[2] The utility-based valuation of Migali has one major drawback, however. It needs to adopt a particular form of utility function; the estimated value of an ICL depends on the form of utility function and the parameters selected.

The goal of the current paper is twofold. First, we propose an alternative way of determining the value of ICLs. Our approach is based on an option-pricing technique, and is complementary to the utility-based valuation of Migali (2010). Whereas the utility-based valuation needs to adopt a particular form of utility function, the option-pricing valuation does not. On the other hand, the option-pricing valuation requires the assumption of the complete market, which is not necessary in the utility-based valuation. The second goal of the current paper is to apply the new technique to the ICLs in Korea and to present a range of estimates for the values of ICLs. Our estimates can be used to assess the magnitude of implicit government subsidy in the ICL program, which may have serious implication on national budget. We hope that our estimates can contribute to the debate of whether

[1] A standard student loan without the income-contingent feature is often called a "mortgage loan." We will not use this term to avoid confusion with a real-estate mortgage loan.

[2] A certainty-equivalent indicates the willingness-to-pay of an individual, not necessarily the market price. Some additional assumptions are required to equate the willingness-to-pay with the market price. One such assumption is that the demand curve for the loans is flat.

the ICL program is an improvement over the SL program. For this reason, we estimate the values of SLs and compare them to the values of ICLs.

Our approach is based on the idea that an ICL contract is equivalent to a series of path-dependent options. The value of each component option can be determined by a numerical option pricing technique. In case of Korean ICLs, our calculation suggests that the implied market value of an ICL contract is about 50% of the nominal value. For example, when the government lends $24,000 to a student through the ICL program, this loan of $24,000 is equivalent to a collection of options with the total market value around $12,000. Put it another way, by choosing terms favorable to borrowers, the government is subsidizing about 50% of the loan. We compared results from the option-pricing valuation to those from the utility-based valuation, and found out that the results are comparable.

The rest of the paper is organized as follows. In Section II, we describe the structure of an ICL contract carefully, and relate it to a series of path-dependent options. In section III, our option-pricing technique is described. Section IV is a detailed description of the ICL program introduced in Korea in 2010. Section V presents the estimates of the value of ICLs in Korea. In Section VI, option valuation and expected utility-based valuation are compared. Section VII concludes the paper.

II. Structure of an ICL Contract

In this section, we describe the structure of an ICL contract, and relate it to path-dependent options. We also analyze an SL contract in the same framework. The reason to analyze an SL is to use it as a benchmark, i.e. to highlight the interesting aspects of an ICL in comparison to an SL.

The borrower of an ICL has an obligation to pay a fixed fraction of his annual income exceeding a predetermined level. If the annual income does not reach the predetermined level, then no payment is made. The obligation disappears either if the entire debt has been paid off or if a certain number of years pass. Suppose that we are at the end of year 0. Let D_0 be the amount of debt outstanding at the end of year 0. Let T be the number of years after which the borrower's obligation expires. Let $K_1,\cdots,K_T$ be the critical income levels, i.e. if annual incomes exceed these levels, payments need to be made. Let ρ be the fraction of the "excess income" to be paid annually; i.e. fraction ρ of the annual income over the critical levels needs to be paid annually. Let r be the interest rate of the loan, i.e. it is the rate at which the amount of debt outstanding grows. The list $(D_0,T,K_1,\cdots,K_T,\rho,r)$ completely specifies the ICL contract at the end of year 0.

Let $Y_1,\cdots,Y_T$ be the annual incomes of year 1 through year T. Let $P_1,\cdots,P_T$ be the payment made at the end of each of year 1 through year T. Let $D_1,\cdots,D_T$ be the amount of debt outstanding after the payment. Note that the value of P is determined after the value of Y is determined. The value of D is determined after the value of P is determined. See Figure 1. The annual payment amount can be expressed as follows:

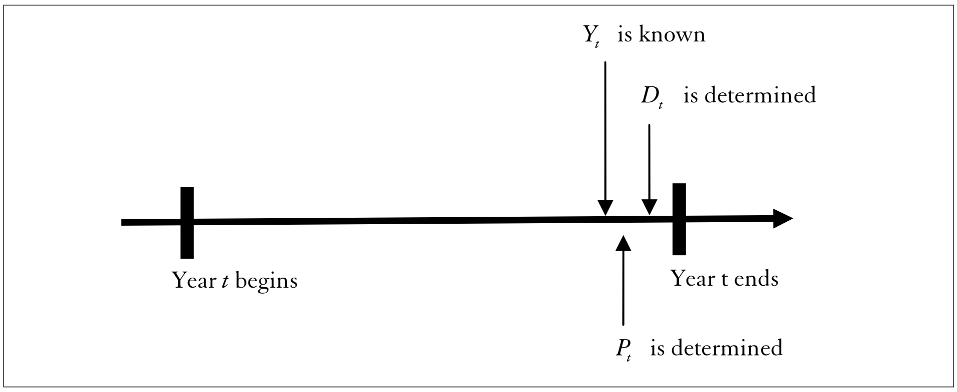
$$\begin{aligned} P_t &= \min\{\max[\rho(Y_t - K_t),0], D_{t-1}e^r\} \\ &= \rho\cdot\max[(Y_t - K_t),0] - \rho\cdot\max\left[\left(Y_t - K_t - \frac{1}{\rho}D_{t-1}e^r\right),0\right] \end{aligned} \tag{1}$$

while the amount of debt outstanding after the payment of P_t is given as:

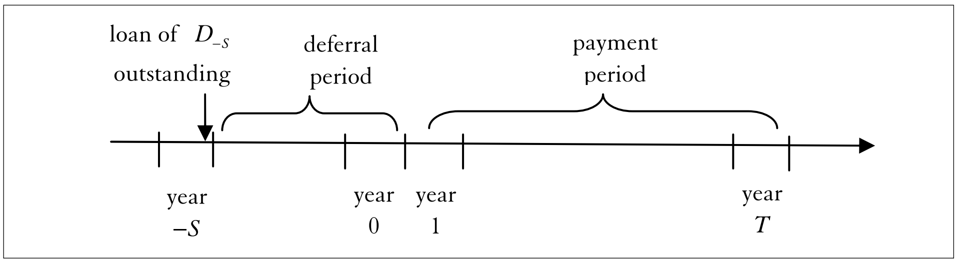
$$D_t = D_{t-1}e^r - P_t \tag{2}$$

It is typical to allow a deferral period of a certain number of years. Suppose that the deferral period is S years. We denote the current year as year $-S$. That is, the amount of D_{-S} is borrowed at the end of year $-S$. Then the timeline looks as in Figure 2.

[Fiugure 1] Timeline



[Figure 2] Timeline with Deferral Periods



From the lender's perspective, an ICL can be considered as a series of path-dependent options. The payment at the end of year t can be considered as a combination of two types of call options. For the first call option, the underlying asset is the income of year t; the strike price is K_t. Let us denote this option by $C(t,K_t)$. For the second call option, the underlying asset is again the income of year t; the strike price is $K_t + D_{t-1}e^r$. Let us denote this option by $C(t,K_t + D_{t-1}e^r)$. Then the payment at the end of year t—to be more precise, holding the right to receive such payment—is equivalent to holding ρ contracts of $C(t,K_t)$ long and ρ contracts of $C(t,K_t + D_{t-1}e^r)$ short. Considering all the annual payments, an ICL is equivalent to the following portfolio:

$$\begin{aligned} ICL = {} & [\rho C(1,K_1) - \rho C(1,K_1 + D_0 e^r)] + \cdots \\ & + [\rho C(T,K_T) - \rho C(T,K_T + D_{T-1}e^r)] \end{aligned} \tag{3}$$

While $C(t,K_t)$ is a standard call option, $C(t,K_t + D_{t-1}e^r)$ is non-standard in that the strike price is a function of D_{t-1}, which is a function of past realizations of *Y*. An option whose payoff is a function of the realized path of the underlying asset is called a path-dependent option.

In a later analysis, we estimate the value of SLs as well as ICLs. While our primary interest is in ICLs, important characteristics of ICLs can be best illustrated in comparison to SLs. An SL can be thought of as a special case of an ICL. While there is no explicit clause waiving the borrowers' obligation to pay in case of low income, the borrowers will be unable to pay if they do not have enough income. Thus, an SL can be thought of as an ICL with a critical income level close to 0.[3] Assuming the constant payment schedule, the relationship between the annual required payment Q_t and the previous year-end debt outstanding D_{t-1} is

$$Q_t + Q_t e^{-r} + \cdots + Q_t e^{-r(T-t)} = D_{t-1}e^r \tag{4}$$

What we are assuming is the following: If there is less than full payment in one year, then the payment of all the remaining years, not just of the next year, is adjusted. (4) implies

$$Q_t = \frac{(e^r - 1)}{1 - e^{-r(T-t+1)}} D_{t-1} \tag{5}$$

The required payment works as a "strike price." The annual payment of an SL can

[3] An SL borrower whose income is almost zero probably has to go through a costly bankruptcy procedure. In that respect, an SL is certainly not identical to an ICL.

be expressed as follows:

$$P_t = \max[(Y_t - K_t),0] - \max[(Y_t - K_t - Q_t),0] \tag{6}$$

where K_t is a critical income level close to 0. In principle, K_t can be zero as the borrower needs to pay even if there is very small amount of income. In reality, however, when income is very small, it is unlikely that the borrower will pay back the loan. The amount of debt outstanding after the payment of P_t is given as:

$$D_t = D_{t-1}e^r - P_t \tag{7}$$

Given (6), the SL can be thought of as a portfolio of long and short positions in call options. Thus, we may write:

$$SL = [C(1,K_1) - C(1,K_1 + Q_1)] + \cdots + [C(T,K_T) - C(T,K_T + Q_T)] \tag{8}$$

III. Applying Option Pricing to ICL

In this section, we explain the option pricing technique that we adopt, and discuss issues arising from the fact that the underlying asset—the borrower's income—is not tradable.

In the previous section, we showed that an ICL contract is a series of path-dependent options. Some popular path-dependent options—lookback options and Asian options—have closed-form pricing formulas. Lookback options are characterized by the payoffs that depend on the maximum or minimum price of the underlying asset during the life of the option. (See Goldman, Sosin, and Gatto (1979) and Conze and Viswanathan (1991).) Asian options are characterized by the payoffs that depend on the average price of the underlying asset during the life of the option. Lookback options and certain types of Asian options have closed-form pricing formulas if they are of the European style.[4] These formulas can be derived from the risk-neutral distribution of the extreme price or the average price. For other types of path-dependent options, a numerical method needs to be adopted. Hull and White (1993) discussed a binomial tree technique. ICL is neither a lookback option nor an Asian option. The payoff $(Y_t - K_t - D_{t-1}e^r)$ is a rather peculiar, non-linear function of past values of Y. Thus, existing closed-form formulas are not applicable, and we need to adopt a numerical approach.[5]

[4] European-style Asian options have closed-form pricing formula when the payoffs depend on a geometric average of the underlying asset.

[5] ICL is certainly not the most complicated path-dependent options that the literature has analyzed.

The derivative product that comes closest to ICL, among those that have been analyzed in the derivative literature, is the mortgage-backed security (MBS) as discussed by Hull and White (1993). MBS is a loan contract with a fixed interest rate, but the borrowers have an option to prepay the principal. They will do so if the market interest rate is lower the MBS rate. Thus, in this case, the underlying index is the market interest rate. Similar to the case of an ICL, the cashflow of MBS depends on the amount of debt outstanding, which is a function of the realized path of the underlying asset. The difference between MBS and ICL is that the payment schedule is specified in the case of ICL but not so in the case of MBS. Also, MBS and ICL have different underlying indexes, and they may have very different distributional characteristics.

Two numerical methods are available for valuing non-standard options: the Monte Carlo method and the binomial method. The Monte Carlo method simulates a path of the underlying index, calculates the payoff from each simulated path, and determines the value of the option from the distribution of the payoffs. The Monte Carlo calculation is "forward moving": at time t of a particular path, the path of the underlying index up to time t is known, but the value of the option is not known. On the other hand, the binomial method generates the "tree" of the underlying index, and calculates the payoff and the value of the option in the backward direction, from the end to the beginning. At each "node" of the tree, the value of the option is known, but the path of the underlying index is not known. In general, there are many paths leading to a particular node. The forward nature makes the Monte Carlo method suitable for path-dependent options for which we need the past path of the underlying index at each point in time. The backward nature makes the binomial method suitable for American options for which we need the value of the live option at each point in time. Neither of them is particularly convenient for path-dependent American options, as discussed by Hull and White (1993) and Thompson (1995). ICL is a European-style path-dependent option. So it is natural to adopt the Monte Carlo approach.

We assume that annual income follows a geometric Brownian motion:

$$\frac{dY}{Y} = \alpha dt + \sigma dZ \tag{9}$$

where α is the instantaneous growth rate of annual income. Under the risk-neutral probability, the data generating process of the annual income series $Y_{-S}, \cdots, Y_0, \cdots, Y_T$ is characterized by

For example, Thompson (1995) discussed take-or-pay contract where payoffs are path-dependent and exercise decisions are made continuously. In ICL, there is no early exercise decision to consider as each component option is of the European style.

$$\log Y_{t+1} \mid \log Y_t \sim \left(\log Y_t + r_f - \frac{1}{2}\sigma^2, \sigma^2 \right) \quad t = -S, \cdots, 0, \cdots, T-1 \tag{10}$$

where r_f is the risk-free rate. We draw from this data generating process and obtain the simulated paths $\{(Y_{-S}^{(1)}, \cdots, Y_T^{(1)}), \cdots, (Y_{-S}^{(J)}, \cdots, Y_T^{(J)})\}$. Using (1) and (2), we obtain the annual payments $\{(P_1^{(1)}, \cdots, P_T^{(1)}), \cdots, (P_1^{(J)}, \cdots, P_T^{(J)})\}$ and the annual debt outstanding $\{(D_1^{(1)}, \cdots, D_T^{(1)}), \cdots, (D_1^{(J)}, \cdots, D_T^{(J)})\}$. The value of the ICL is the expected value of the discounted annual payments, i.e.

$$V_{-S} = \frac{1}{J} \sum_{j=1}^{J} (P_1^{(j)} e^{-r_f(S+1)} + \cdots + P_T^{(j)} e^{-r_f(S+T)}) \tag{11}$$

It is possible to modify the formula to improve the accuracy of the above calculation somewhat.[6] One can also quantify the accuracy of the calculation in the following way. Noting that V_{-S} is a sample mean, its standard deviation is the population standard deviation divided by the square root of the sample size (J). The population standard deviation can be estimated by the sample standard deviation. Thus, the standard error for V_{-S} is

$$se(V_{-S}) = \frac{1}{\sqrt{J}} \sqrt{\frac{1}{J} \sum_{j=1}^{J} (P_1^{(j)} e^{-r_f(S+1)} + \cdots + P_T^{(j)} e^{-r_f(S+T)} - V_{-S})^2} \tag{12}$$

The application of the risk-neutral pricing technique as described above requires two assumptions: First, income is a log-normal diffusion process as expressed in (9). Second, capital market is complete so that the income risk can be completely hedged by a combination of available securities, i.e. a riskless hedging portfolio can be created. Below we discuss each of these two assumptions in turn.

The first assumption is often made in the literature, and it appears to be accepted as a convenient approximation. Hogan and Walker (2007) make this assumption in

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[6] (11) can be modified as

$$V_{-S} = \sum_{t=1}^{T} \rho e^{-r_f(S+t)} E^*(\max[(Y_t - K_t), 0])$$
$$-\rho e^{-r_f(S+t)} E^* \left(\max\left[\left(Y_t - K_t - \frac{1}{\rho} D_{t-1} e^{r_f} \right), 0 \right] \right)$$

where E^* is the average over j. Let $W(a,b,c)$ be the Black-Scholes call option formula where a is the current value of the underlying asset, b is the strike price, and c is the time to maturity. Then we may rewrite (11) as:

$$V_{-S} = \rho \sum_{t=1}^{T} W(Y_{-S}, K_t, S+t) - \rho \sum_{t=0}^{T-1} e^{-r_f(S+t)} E^* \left[W \left(Y_t, K_{t+1} + \frac{1}{\rho} D_t e^{r_f}, 1 \right) \right]$$

their model of education choice; Venegaz-Martinez (2006) in his study of dynamics of the exchange-rate and labor income; Koo (1998) in a portfolio selection problem; and Hu (1993) in analyzing taxation. Migali (2010) also makes this assumption in his study of ICL. Two short-comings are worth mentioning, though. First, a diffusion model requires that annual average incomes continue to grow over time, while labor market data suggest that annual average incomes peak at a certain age. Falling annual average incomes are inconsistent with a diffusion model as it would imply negative interest rates.[7] Second, a simple diffusion model such as (9) assumes that annual incomes are always positive, inconsistent with the real world situation where unemployment may bring annual (labor) incomes close to zero. In a more general model such as Poisson-normal process, annual incomes can be allowed to be zero. While we could attempt to work with such a model, the gains from doing that seem rather limited. Whether annual incomes are zero or small positive numbers is less significant than whether annual incomes are above or below payment-triggering critical income level K_t. Given the limited gains from using a more complicated model, we work with a simpler model of (9).

The second assumption of the complete market is quite strong. It is unlikely that one can create a riskless hedging portfolio out of existing securities. When a riskless hedging portfolio cannot be created, there is no guarantee that the price suggested by the risk-neutral pricing technique is "correct." Despite these limitations, we believe that risk-neutral pricing can be useful in the valuation of ICLs for two reasons. First, it provides a non-arbitrary benchmark, which is independent of the choice of a particular utility function. Moreover, the results obtained from the risk-neutral pricing are quite reasonable. We show later that the solution obtained from the risk-neutral pricing is well within the range of solutions that one can obtain by assuming utility functions with conventional values of parameters. Second, the solution of a utility-based valuation is not a good indicator of an equilibrium price. Only under special circumstances, the utility-based valuation can be interpreted as an equilibrium price. The relationship between the option-based value and the utility-based value is further discussed in Section VI.

IV. Estimation of the Value of ICLs in Korea

1. ICL Program in Korea

The nation-wide ICL program for university students was introduced in 2010 in Korea. The program was administered by Korea Student Aid Foundation (KOSAF),

[7] A diffusion model allows the possibility of an individual's income to fall; it does not allow the possibility of an individual's expected income to fall.

a government-funded agency in charge of all student-loan and scholarship programs.[8]

In the first half of 2010, about 100,000 students took ICLs from KOSAF. The total amount borrowed was 424 million US dollars.[9] This is about 30% of the total lending of KOSAF. KOSAF plans to phase out all existing SL programs, and plans to offer only ICLs to university students eventually.

On average, a student borrowed about \$4,240 per semester. If a student borrows this amount every semester for 4 years, the total borrowing would amount to \$4,240 x 8 = \$33,290 plus the interest accrued between the initial and the last borrowings.

The ICL in 2010 had the following characteristics:

- The critical income level (K) for year 2010 was \$16,320. It was based on the minimum 4-person household income as determined by Ministry of Health, Welfare, and Family Affairs.
- The fraction of excess income to be paid (ρ) was 0.2.
- The annual interest rate of the loan (r) was 4.5%.
- The borrower's obligation expires when the borrower becomes 65 years old. Thus, the payment period (T) can be as low as 35 years (if the borrower graduates the university at the age of 30) or as high as 40 years (if the borrower graduates the university at the age of 25).

2. Estimation of the Value of ICLs in Korea

Now we present the estimated value of a typical ICL contract offered to Korean students in 2010. We also estimate the value of a comparable SL contract. We first describe the parameters, and then we describe the estimates.

We consider a male who is graduating a 3-year college at the end of year 2010. We consider a male borrower rather than a female borrower. Female labor-market participation rate is significantly lower than male participation rate, and this makes the valuation problem more difficult for female. There is no 3-year college in Korea in reality; there are only 2-year colleges and 4-year colleges. We are assuming a fictitious average case. We do not have separate income estimates for 2-year and 4-year college graduates; the income estimates that we have from the study by Lee and Jeon (2009) are for the average case.

We assume the borrower starts paying back the student loan at the end of year 2014. The deferral period is 3 years and the payment period is 35 years. The 3-year deferral period reflects the mandatory military service. As mentioned earlier, the

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[8] The information in this section was collected from various publications from KOSAF, many of which are available in its website (www.kosaf.go.kr).

[9] For convenient conversion between Korean won and US dollar, we assume that the exchange rate of 1,000 Korean wons per 1 US dollar throught this paper.

payment obligation expires when the borrower reaches the age of 65. Thus, we are assuming that the borrower is 30 years old when he finished his university education and his military service.

We chose initial income Y_{-S}, the mean growth rate of income α, and the variance of income σ by solving the following equations:

$$E(Y_{2014}) = Y_{-S}e^{4\alpha}$$
$$E(Y_{2048}) = Y_{-S}e^{38\alpha}$$
$$\sqrt{V(Y_{2048})} = Y_{-S}e^{38\alpha}\sqrt{e^{38\sigma^2}-1} \tag{13}$$

$E(Y_{2014})$ and $E(Y_{2048})$ are the mean incomes in year 2014 and year 2048 of those who graduated universities at the end of year 2010. $V(Y_{2048})$ is the variance of income in year 2048 of those who graduate universities at the end of year 2010. The above equations come from the mean and the variance formulas of log-normal distribution and from the fact that year –*S* corresponds to year 2010.[10] We chose year 2014 and year 2048 as they are the first and the last years of the payment period. The values for $E(Y_{2014})$, $E(Y_{2048})$, $V(Y_{2048})$ are from Lee and Jeon (2009) whose estimation is based on Korean labor panel data. Estimated values of (Y_{-S}, α, σ) are (16068,.0578,.1485).[11]

We obtain an alternative set of estimates. Estimated mean income declines from year 2042, making our Y_{-S} estimate potentially too high and our α estimate potentially too low. We obtain an alternative set of parameters using year 2041 data, i.e.

$$E(Y_{2014}) = Y_{-S}e^{4\alpha}$$
$$E(Y_{2041}) = Y_{-S}e^{31\alpha}$$
$$\sqrt{V(Y_{2041})} = Y_{-S}e^{31\alpha}\sqrt{e^{31\sigma^2}-1} \tag{14}$$

The new estimates of (Y_{-S}, α, σ) are (14424,.0848,.1311).

Below is the description of our parameter selection:

- We set Y_{-S} to be 16,068 from (13).
- We vary the value of σ to be .05, .10, and .15. These numbers reflect the estimates obtained from (13) and (14). Migali (2010), reflecting the conditions in UK labor market, chose the value σ to be 0.02, 0.05, 0.15. Venegas-

[10] Note that σ appears both in the mean and the variance parts of (10); two σ's cancel each other and does not appear in the first two equations of (13).

[11] Note that the value of α does not influence the value of ICL. At this point, we are mainly interested in choosing a representative value of Y_{-S}.

Martinez (2006) set the value of σ as 0.2, but this value appears to be unrealistically high. Our baseline value of σ is .10.[12]

- The value of critical income K_{-S} is \$16,320. This is the actual number as described in the previous section. K grows at the annual rate of 3.5%. This is the average growth rate of the minimum 4-person household income from 2005 to 2010 in Korea. See Ministry for Health, Welfare, and Family Affairs (2009).
- The value of payment rate ρ is 0.2. This is the actual figure as described in the previous section.
- The value of loan interest rate r is 0.045. This is the actual figure as described in the previous section.
- The amount of debt D_{-S} was set to be \$24,000. A typical student borrows about \$4,000 per semester as mentioned in the previous section. Borrowing this amount for 3 years (6 semesters) would result in \$24,000. We ignore the interest accrued since the initiation of the loan.

For SLs, we use the identical parameters except for the critical income. As there is no set critical income for SLs, we use one-fourth of the ICL critical income. For all the computations below, we set the simulation size J to be 1,000,000.

[Figure 3] Present Value of Yearly Payments ($\sigma = .10$)

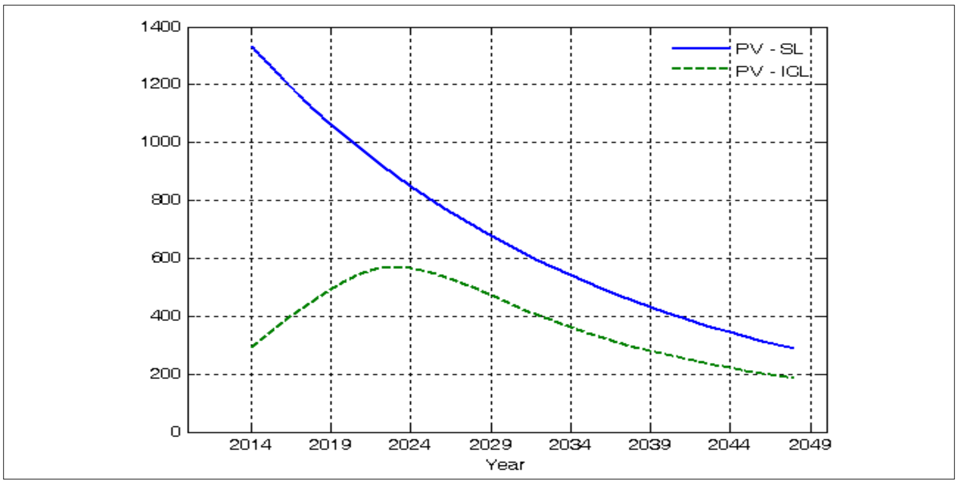

Figure 3 plots the present value of annual payments, each of T terms in (11), for the SL and for the ICL. The area under the curve is the option-pricing value of the SL and the ICL. The volatility was chosen to be 10%. By construction, the annual payments for the SL are constant and the discounted present values are declining over years. For the ICL, annual payments are increasing initially but after about 8 years they stabilize. Thus, the discounted present value of the ICL payment is

[12] The baseline value is the one that we believe is the most realistic.

decreasing after about 8 years. The figure shows clearly that the advantages of the ICL over the SL to the borrowers are greatest in earlier years, when the income level is low relative to the payment triggering income level. The present value of SL payment is always higher than the present value of ICL payment. This is due to the fact that there is a significant probability that the ICL borrower may not be able to pay back the loan completely by year 2048. Setting the interest rate of SL identical to the interest rate of ICL makes ICLs much cheaper. While not shown here, when the volatility is higher, initial payments are larger and the present value of later payments becomes smaller. The mode of the distribution moves to an earlier year. When the volatility is smaller, the mode of the distribution moves to a later year.

[Table 1] Option-Pricing Value of SLs and ICLs

The amount of debt outstanding at the end of year 2010 is $24,000. The baseline case, which corresponds to Figure 3, is indicated by the asterisk. Numerical standard errors are inside parenthesis.

σ	SL		ICL		SL-ICL	
0.05	24,000.00	(0.00)	15,093.29	(9.11)	8,906.71	
0.1	23,976.46	(0.34)	13,365.75	(10.16)	10,610.71	*
0.15	23,541.05	(1.88)	12,395.73	(10.47)	11,145.32	

Table 1 shows the option pricing value of SLs and ICLs for different values of volatility parameter σ. Recall that $D_{-S} = 24000$. Thus, what the table suggests is: the true value of $24,000 SL loan is somewhere between $23,000 and $24,000, while the true value of $24,000 ICL loan is somewhere around $13,000. As the volatility rises, the value of ICL declines, reflecting larger insurance value. For the same reason, the difference between SL and ICL increases as the volatility rises.

V. Comparing Option Valuation to Expected Utility-Based Valuation

As mentioned earlier, the risk-neutral pricing technique assumes that a riskless portfolio can be created out of existing securities, which is possible if capital market is complete. The requirement of complete market is unlikely to be satisfied in reality, and the valuation based on the risk-neutral pricing may deviate from the true valuation. We provide an alternative valuation based on the concept of certainty equivalent. We do not assume complete market, but we assume a particular type of utility – additively separable, constant relative-risk-aversion (CRRA) utility. CRRA utility is used by other studies including Venegas-Martinez (2006), Hogan and Walker (2007), and Migali (2010).

The certainty equivalent value of ICL can be calculated as follows: First, we calculate the expected utility of an individual who took ICL. Let us call the level of expected utility in this case U^{ICL}. Second, we calculate the expected utility of the same individual assuming that the individual did not take ICL. Let us call the expected utility in this case U^0. Because of consumption smoothing effect, a loan tends to increase the utility. So, we are likely to have $U^{ICL} > U^0$. The third step of the calculation is to find out the amount of cash compensation that is necessary to make U^0 equal to U^{ICL}.[13]

The certainty equivalent is typically measured as the amount of cash payable now, rather than cash payable sometime in the future. Calculating certainty equivalent this way creates a problem for us, as we are dealing with students who do not have any current income. For a risk-averse individual, cash of $100 means much more when the income level is lower. So we may end up over-estimating the certainty equivalent value of an ICL. We measure the certainty equivalent as the amount of cash payable now pretending that Y_{-S} is the income of year –*S*. Recall that Y_{-S} is not a true income in year –*S*. Y_t is true income only for $t \geq 1$.

We assume an additively separable, von Neuman-Morgenstern expected utility:

$$U(X_{-S}, X_{-S+1}, \cdots) = E\left[\sum_{t=S+1}^{\infty} e^{-\delta t} u(X_{-S+t})\right] \tag{15}$$

where X_t is the consumption during year t. The individual is assumed to live infinitely many years. Note that this assumption is without any consequence; our analysis is not affected by consumption after year *T*. The borrower does not have any income until the payment period starts. i.e. $X_{-S} = \cdots = X_0 = 0$. Rather than arbitrarily choosing the level of consumption for this period, we consider the utility starting from year 1 only. We ignore the problem of intertemporal consumption decision, i.e. we assume that individuals consume all their incomes each year. Thus, consumption equals the income after debt payment. As in the previous section, income is assumed to be a normal diffusion process. The difference, however, is that we now need to examine the “real” distribution of income, not the risk-neutral distribution of income. Thus, (10) should be replaced with the following:

$$\log Y_{t+1} \mid \log Y_t \sim \left(\log Y_t + \alpha - \frac{1}{2}\sigma^2, \sigma^2\right) \quad t = -S, \cdots, 0, \cdots, T-1 \tag{16}$$

We denote the ICL payment of year t by P_t^{ICL}. Then the certainty equivalent value of ICL, X^*, is given by

[13] See Brown (2001) for applying the certainty equivalent idea to the valuation of an annuity.

$$\begin{aligned} &u(Y_{-S}) + E[e^{\delta(S+1)}u(Y_1 - P_1^{ICL}) + \cdots + e^{\delta(S+T)}u(Y_T - P_T^{ICL})] \\ &\quad = u(Y_{-S} + X^*) + E[e^{\delta(S+1)}u(Y_1) + \cdots + e^{\delta(S+T)}u(Y_T)] \end{aligned} \tag{17}$$

A similar formula can be written down for the certainty equivalent of an SL.

For instantaneous utility $u(\ )$, we use the CRRA utility function, i.e.

$$u(X) = \frac{X^{1-\gamma}}{1-\gamma} \tag{18}$$

where γ is the relative risk aversion.

The standard error for X^* can be calculated in the following way. Noting that X^* is a non-linear function of a sample mean, we draw from the distribution of the sample mean, and simulate the distribution of X^*. If $Z^{(1)},\cdots,Z^{(J)}$ is a sample, then $\overline{Z} + \frac{1}{\sqrt{J}}(Z^{(1)} - \overline{Z}),\cdots,\overline{Z} + \frac{1}{\sqrt{J}}(Z^{(J)} - \overline{Z})$ can be considered draws from the distribution of the sample mean $\overline{Z}$. From this, we can simulate the distribution of any function of $\overline{Z}$.

Recall that option-pricing valuation is derived from the no-arbitrage condition. If the market is complete such that an ICL can be replicated by a portfolio of other securities, then the price of the ICL should be identical to the price of the replicating portfolio. The no-arbitrage condition guarantees that there will be no other equilibrium price in the market. On the other hand, utility-based valuation does not provide an equilibrium price. Rather, its solution can be interpreted as the reservation price. If the utility-based value of an ICL is \$20,000, it means that the borrower is willing to accept the payment obligation if the loan amount equals to or exceeds \$20,000. There is no guarantee that the equilibrium price in the market should be \$20,000. Of course, if the quantity (i.e. the amount of loans that we used in the valuation) happens to be the equilibrium quantity, the reservation price of the representative agent will be the equilibrium price. However, the existence of a representative agent and the assumption of equilibrium quantity are two very strong assumptions.[14]

[14] If the market is complete, the option-pricing value of an ICL is the market price. However, if the market is not complete, the option-pricing value of an ICL can be lower than the market price. The demand for an ICL can be high if the ICL provides a way for an individual to hedge against previously unhedgeable risk, resulting in a higher market price than implied by complete-market option-pricing approach. (If a new insurance product offers protection against a previously unprotectable event, then the price of this product can be quite high, i.e. higher than when similar products were available previously.) The utility-based value reflects reservation price, and it does not necessarily indicate the market price. However, under the special circumstance mentioned above (i.e. the quantity happens to be the equilibrium quantity) we can state: if the market is incomplete, then the option-pricing value can be lower than the utility-based value. Our calculation shows that the option-pricing value is not very different from the utility-based value. One possible interpretation is that the incompleteness of the market has limited effect on the market price.

For the evaluation of (17), risk aversion γ and time perference δ as well as mean growth rate α need to be specified. Hogan and Walker (2007) set the value of γ to be one, i.e. $\lim_{\gamma\to 1}\frac{X^{1-\gamma}-1}{1-\gamma}=\log(X)$, and the value of δ to be 0.1. Migali (2010) report the calculation based on the γ value of 0.25, 0.5, and 1.2, the δ value of 0.08, 0.15, 0.3, and the α value of 0.01. Venegas-Martinez (2006) set the value of δ as 0.08, γ to be one, and α to be 0.02. We repeat our calculation for each value of $\gamma = 0.25, 0.5, 1.0$. The value of δ is 0.045 so that it corresponds to the interest rate. The value of α are .055, .07, .085, reflecting the estimates from (13) and (14).[15] All other parameters are as explained in the previous section.

[Figure 4] Expected Utility After Yearly Payments $(\sigma=.1, \alpha=.055, \gamma=.5)$

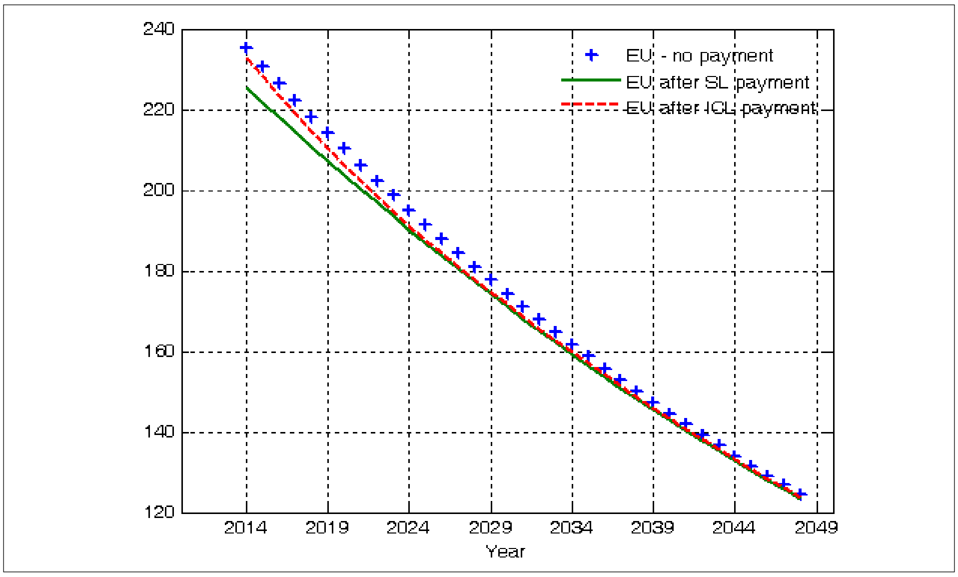

Figure 4 plots the expected utility of each of T years in (17). Naturally, the expected utility is highest when there is no payment; the expected utility after SL payments is lower than the expected utility after ICL payments. SL payments are higher than ICL payments on average.

Table 2 shows the utility-based value of SLs and ICLs for various combinations of parameter values. In the baseline case of $\gamma=.5$, $\sigma=.1$, $\alpha=.055$, the SL value is \$21,200, and the ICL value is \$12,200. These values are comparable to what we obtained from the option pricing in the earlier analysis.

[15] Our value of α is substantially higher than those of Migali (2010) and Venegas-Martinez (2006); Income growth rate is much higher in Korea than in a typical developed economy; the risk-free rate of 0.045 is consistent with this fast growth rate.

[Table 2] Utility-Based Value of SLs and ICLs

The amount of debt outstanding at the end of year 2010 is $24,000. The baseline case, which corresponds to Figure 4, is indicated by the asterisk. Numerical standard errors are inside parenthesis.

γ	σ	α	SL		ICL		SL-ICL	
0.25	0.05	0.085	19,414.95	(0.84)	21,262.16	(0.36)	-1,847.21	
		0.07	20,659.52	(0.91)	21,064.65	(0.96)	-405.13	
		0.055	22,021.86	(0.99)	18,643.91	(5.05)	3,377.95	
	0.1	0.085	19,796.94	(1.73)	20,528.99	(2.20)	-732.05	
		0.07	21,076.64	(1.88)	19,009.16	(5.03)	2,067.48	
		0.055	22,478.01	(2.04)	14,960.48	(8.12)	7,517.53	
	0.15	0.085	20,444.75	(2.70)	18,689.51	(5.20)	1,755.24	
		0.07	21,726.89	(2.86)	16,331.16	(7.42)	5,395.73	
		0.055	22,968.81	(2.97)	12,798.09	(8.85)	10,170.72	
0.5	0.05	0.085	15,736.38	(1.39)	18,432.43	(0.64)	-2,696.05	
		0.07	17,761.93	(1.62)	18,130.41	(1.31)	-368.48	
		0.055	20,187.55	(1.92)	15,612.77	(4.75)	4,574.78	
	0.1	0.085	16,472.10	(2.97)	17,360.43	(2.43)	-888.33	
		0.07	18,641.66	(3.48)	15,888.07	(4.79)	2,753.59	
		0.055	21,247.61	(4.12)	12,291.53	(7.18)	8,956.08	*
	0.15	0.085	17,780.66	(4.91)	15,392.07	(4.85)	2,388.59	
		0.07	20,129.46	(5.60)	13,330.65	(6.59)	6,798.81	
		0.055	22,690.59	(6.12)	10,374.52	(7.58)	12,316.07	
1	0.05	0.085	10,528.88	(1.86)	13,004.15	(0.92)	-2,475.27	
		0.07	13,166.47	(2.51)	12,637.12	(1.66)	529.35	
		0.055	16,941.48	(3.54)	10,375.11	(3.84)	6,566.37	
	0.1	0.085	11,759.75	(4.41)	11,680.10	(2.47)	79.65	
		0.07	14,918.49	(6.10)	10,514.94	(3.93)	4,403.55	
		0.055	19,545.73	(8.79)	7,991.46	(5.18)	11,554.27	
	0.15	0.085	14,349.46	(8.80)	9,867.82	(3.84)	4,481.64	
		0.07	18,468.66	(11.92)	8,499.60	(4.77)	9,969.06	
		0.055	24,181.23	(15.93)	6,589.45	(5.18)	17,591.78	

As the parameter values vary, the values of the ICL and the SL vary as well. The following observations can be made:

- Higher risk aversion (i.e. higher γ) lowers the value of the ICL as risk-averse individuals would consider the ICL less burdensome. A higher risk aversion lowers the value of the SL as well as the value of the ICL, so the effect on the difference between the SL and the ICL is not uniform.
- Higher volatility (i.e. higher σ) lowers the value of the ICL. Insurance feature of the ICL is more valuable when income volatility is higher. The effect of volatility on the value of SL is not as great; thus, the difference between SLs

and ICLs grows with volatility.

- Higher income growth rate (higher α) increases the value of the ICL. With higher income, the payment amount and the payment probability are higher as well, making the ICL more valuable. At the same time, the SL payment becomes smoother, making the relative merit of ICLs smaller. Thus, the difference between SLs and ICLs diminishes as income growth rate becomes larger.

VI. Conclusion

Proceeding from the idea that an ICL contract is a series of path-dependent options, we computed the approximate value of an ICL contract offered to Korean students in 2010. Our calculation suggests that, when the government lends \$24,000 to a student through an ICL program, the market value of this loan is around \$12,000.

Option pricing requires a number of strong assumptions including the tradability of the underlying asset. We estimated the value of an ICL contract using an alternative approach, assuming an expected-utility with certain risk aversion. While the estimates depend on the choice of parameter values, the option-based estimates are reproduced with a reasonable set of parameters.

While our calculation is only approximate, it is useful in that the implicit government subsidy in ICL can be assessed. Our calculation suggests that up to half of the loan amount is subsidized by the government through a combination of the low interest rate and the insurance feature of the ICL.

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