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A Test for Trading Time Hypothesis on Weekends under Time Varying Autoregression with Heteroskedasticity*

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Standard daily financial time series analyses using autoregressive (AR) models typically disregard weekends following the trading time hypothesis (TTH) because the relevant assets of the models are not traded (and thus, their prices are not observed) on weekends. However, weekends may affect asset prices through time discounting as well as through shocks/news occurring on weekends. In this regard, we suggest a test for the TTH by using an AR(1) model, where many asset prices are closely approximated by an AR(1) process. The proposing test statistics are based upon the differences of AR coefficients and error variances between Monday and the other weekdays. Asymptotic normality of the suggested test statistics under the TTH and model stationarity is proved. Under the model of nonstationarity, the test statistic is asymptotically pivotal/non-standard and the critical values are given from the Monte Carlo simulations. In an application for the United States S&P 500 data during the years 2000-2011, we found that the TTH was rejected, particularly during the years of war and financial crisis. We also confirmed a weakening of the weekend effect as depicted in Chow, Hsiao and Solt (2003), and Connolly's (1989) results. It requires us to revise the dynamic analyses using a time series model of asset prices considering the weekends.

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I. Introduction

Typical analyses of daily asset prices (e.g., stock prices, exchange rates, interest rates) data omit weekend data simply because the market does not open on weekends and therefore, assets are not traded (and their prices are not observed).

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Hence, typical autoregressive (AR) or vector autoregressive (VAR) time series analyses of asset prices, including regressions and forecasting, obscure the role of weekends. That is, Friday is usually assumed to be the last day of the week, and Monday is considered to be contiguous to Friday as Monday is to Tuesday. This is often referred to as the trading time hypothesis (TTH) for weekends and is opposite to the calendar time hypothesis (CTH), which assumes that the same model is applied to weekends as well as to weekdays. The TTH has the following theoretical implications for the model: First, no shocks occur during weekends. Second, time discounting of the past shocks to the current asset prices does not occur during weekends; therefore, a shock to the asset prices on Friday influences those on Monday with no forgetting as a shock to the asset prices on Monday influences those on Tuesday.

However, we suspect that older shocks are less likely to influence asset prices than newer shocks. That is, time discounting due to the forgetting during weekends may matter, particularly for weekends. That is, a shock occurring on a Friday influences Monday prices after a two day delay over the weekend, whereas that occurring on a Thursday influences Friday prices without any delay. Further, we suspect that shocks to the asset prices may arise during the weekends as well as during the weekdays. For instance, wars affecting the asset prices find no breaks during weekends and governments can announce a bailout policy on any day of the week, including weekends, during economic crises.

There are considerable evidences/literatures that support the hypothesis that weekends influence weekday asset prices. For instance, it is well known that Monday stock returns (computed from the previous week's Friday closing price) are lower than those of the other weekdays (the weekend effect). Fields (1931), Cross (1973), French (1980), Rogalski (1984), and Harris (1986), among others, confirm the weekend effect on stock markets. Gibbons and Hess (1981) discover similar weekend effects from the Treasury bill data. From a different perspective, Tsiakas (2004) argues that information accumulated over the weekends (and holidays) is a predictor of subsequent daily volatility. French and Roll (1986) find that stock prices are much more volatile during trading hours than during non-trading hours, including weekend hours.

Noteworthy is that any misspecification error may occur from the inconsistent estimation of the AR coefficient when the TTH is not admitted. Obviously, it invalidates the conventional dynamic analyses, including impulse response analysis and optimal forecasting.

In this regard, we suggest a test for the TTH by using an AR(1) model, where many asset prices are closely approximated by an AR(1) process. According to Fama's (1965) efficient markets argument, asset price changes are unforecastable. Hall (1978) argues that consumption should satisfy the random walk process. Barro (1979) and Mankiw (1987) develop related arguments for taxes levied and new money issued by central banks. Changes in foreign exchange rates are argued to be unpredictable as well (see Diebold and Nason, 1990). From this fundamental observation, Litterman (1986) and Doan, Litterman and Sims (1984) suggested a Bayesian VAR forecasting.

The proposing test statistics are based upon the differences of AR coefficients and error variances between Monday and the other weekdays. Asymptotic normality of suggested test statistics under the TTH and model stationarity is proved. Under the model nonstationarity, the test statistic is asymptotically pivotal/non-standard and the critical values are given from the Monte Carlo simulations. In an application for the United States S&P 500 data during the years 2000-2011, we found that the TTH was rejected, particularly during the years of war and financial crisis. We also confirmed a weakening of the weekend effect, as discovered by Chow, Hsiao and Solt (2003), and Connolly's (1989) results.

The rest of the paper is organized as follows: Section 2 introduces a time varying autoregression for the weekend. Section 3 introduces the test statistics for the TTH. Section 4 presents the simulation and applications, and Section 5 concludes. Standard notations are used throughout the paper without explicit references. In particular, we use $\xrightarrow{p}$ and $\xrightarrow{d}$ to indicate convergence in probability and distribution, respectively.

II. Autoregression with Time-varying Coefficients and Unobservable Variables

Consider the following time-varying coefficient autoregressive model with the heteroskedasticity:

$$P_t = \theta_t P_{t-1} + \delta_t u_t \tag{1}$$

where P_t is a demeaned/detrended (or log-transformed) asset price; and $\{u_t;\ t \in \mathbb{Z}^+\}$ is a mean zero i.i.d process with variance $\sigma^2 > 0$. Specifically, for modeling and data observations, we assume that Model (1) operates at fixed time intervals normalized to unity. In each basic period, the process generates a value. Here P_t may be observed repeatedly at regular intervals (e.g., 24 hours) except for the weekends (see Table 1).[1]

Note that Model (1) may be readily generalized to a vector autoregressive (VAR, 1) model when P_t is a vector. Further, note that an asset price (e.g., stock price) is theoretically a function of the fundamentals (e.g., dividend). Thus, the asset price

[1] For stock markets, a weekend spans from the Friday close to the Monday open.

may be changed because the fundamental may be changed during the weekends. For instance, Starbucks outlets open during weekends, and thus, their profits are allocated to weekdays. Clearly, this influences the dividends.

[Table 1] Samples with weekends as missing observations

	Mon	Tues	Wed	Thur	Fri	Sat	Sun
observavbiltiy	o	o	o	o	o	x	x
t	k+1	k+2	k+3	k+4	k+5	k+6	k+7

Note: 1) o denotes observable variables, and x, unobservable variables.
2) The k(=0,1,2,...,n-1) denotes the number of week.

There are two important differences between Model (1) and the periodic autoregression in Bentarzi and Hallin (1996) and McLeod (2008). First, Model (1) includes the heteroskedasticity for the error variance. Second, an asset price P_t is not observed during the weekends in Model (1) (see Table 1).

Accordingly, we denote the sets for the days of a week as $D_W \equiv$ {Saturday, Sunday}, $D_M \equiv$ {Monday}, $D_T \equiv$ {Tuesday} and $D_{TF} \equiv$ {Tuesday, Wednesday, Thursday and Friday}. The other holidays[2] are not simultaneously considered in this paper because other holidays may have different sizes of shock and time discounting during the holidays. It is well recognized that Christmas and Thanksgiving Day have often different effects to the stock prices than other non-weekend holidays; Santa Claus rally[3] or Thanks giving rally are such examples.

Now we remind ourselves that the TTH is directly stated as

$$P_t = \theta P_{t-3} + u_t \quad \text{if} \quad t \in D_M \quad \text{(under TTH)}, \tag{2}$$

and thus, Monday is considered to be contiguous to Friday as Monday is to Tuesday:

$$P_t = \theta P_{t-1} + u_t \quad \text{if} \quad t \in D_{TF}\,. \tag{3}$$

Now we suppose that all equations for weekdays have the same coefficients and error variances. However, the weekend equations may have different AR coefficients and error variances, reflecting the shocks, which occur during weekends, and time discounting:

[2] Holidays, such as New Year's Day, Martin Luther King, Jr. Day, Independence Day, Labor Day, Columbus Day, Veterans Day, Thanksgiving Day and Christmas for the U.S., may be similarly analyzed using the developed method.

[3] This rally is often attributed to the anticipation of the (following) January injection of funds into the stock market, and to the increase of trades which must, for tax reasons, be completed by the end of the year.

Assumption 2.1

$$\begin{aligned}\theta_t &= \theta \ \ and \ \ \delta_t = 1 \ \ if \ \ t \in D_M \cup D_{TF} \\ &= \theta_W \ \ and \ \ = \delta \ \ if \ \ t \in D_W\end{aligned}$$

where $0 < \theta, \theta_W \leq 1$, *and* $0 \leq \delta \leq 1$.

Then we check how the TTH imposes the restrictions to the model coefficients for weekends (θ_W or δ). Suppose the TTH is not imposed. Then if t is in D_M, then the P_t can be written as

$$P_t = \theta_M P_{t-3} + u_{Mt} \ \ \text{if} \ \ t \in D_M ; \tag{4}$$

after a repetitive substitution for unobservable prices during weekends (P_{t-1}, $P_{t-2} \in D_W$), where

$$\theta_M = \theta\theta_W^2 \tag{5}$$

and

$$u_{Mt} = \theta_W \theta \delta u_{t-2} + \theta \delta u_{t-1} + u_t . \tag{6}$$

Under the TTH, equations (2) and (4) are equivalent models if we impose $\theta_W = 1$ and $\delta = 0$. Hence, we may reinterpret the TTH (or CTH) as:

Proposition 2.2 *Suppose that* $\theta_W = 1$ (*or* θ) *and* $\delta = 0$ (*or* 1). *Then the TTH* (*or the CTH*) *holds*.

Note that $\theta_W = 1$ implies that there is no time discounting over the weekend, and $\delta = 0$ implies that no shock occurs during weekends. According to French and Roll (1986), stock prices are much more volatile during trading hours than during non-trading hours, including the weekend hours. If this result is admitted, then we expect that $\delta < 1$ (and the error variance of weekends is smaller than that of weekdays).

The smaller θ_W compared to $\theta(\theta_W < \theta)$ implies that time discounting of the Friday shocks to Monday asset prices may arise more largely during weekends than during weekdays.[4] Ebbinghaus (Memory: A contribution to experimental psychology, 1885, http://psychclassics.yorku.ca/Ebbinghaus/index.htm) extrapolated

[4] During weekends, we go to church, visit an amusement park, take a trip, go to a movie and attend a wedding ceremony. So forgetting may arise more rapidly during weekends than weekdays because our life style is changed from that of the weekdays.

the hypothesis of the exponential nature of forgetting: $R = e^{-t/s}$ where R is memory retention, s is the relative strength of the memory, and t is time. Thus our case θ_W^t is a special case of Ebbinghaus if we let $\theta_W = e^{-1/s}$ where t is the number of days (t=2 for weekend and t=1 for weekday). Therefore, the size of θ_W may be an index reflecting the memory retention as time passes in an asset market.

Finally, the null hypothesis for testing TTH becomes

$$H_0 : \theta_W = 1 \text{ and } \delta = 0. \tag{7}$$

If Model (1) is a VAR (1) where P_t is a $\ell \times 1$ vector (e.g., a bivariate model with interest rate and stock price), then the above TTH may be transformed as $\theta_W = I_\ell$ and $\delta = 0_\ell$ where I_ℓ is an ℓ dimensional identity matrix and 0_ℓ is a $\ell \times 1$ zero vector. In the VAR case, the off-diagonal elements of θ_W are all zeros, which is an additional restriction not appearing in the univariate AR model.

What will happen if the TTH does not hold? Following remark addresses this issue.

Remark 2.3 First, a conventional OLS (ordinary least square) estimator using all weekday data (directly connecting Monday and the previous Friday) asymptotically converges to a weighted average of θ and θ_M :

$$\frac{\sum_{D_M \cup D_{TF}} P_{t-1}P_t}{\sum_{D_M \cup D_{TF}} P_{t-1}^2} = \hat{\theta}\frac{\sum_{t\in D_{TF}} P_{t-1}^2}{\sum_{D_M \cup D_{TF}} P_{t-1}^2} + \hat{\theta}_M \frac{\sum_{t\in D_M} P_{t-1}^2}{\sum_{t\in D_M \cup D_{TF}} P_{t-1}^2} \xrightarrow{p} \theta\lambda + \theta_M(1-\lambda).$$

where $\sum_{t\in D_{TF}} P_{t-1}^2 / \sum_{D_M \cup D_{TF}} P_{t-1}^2 \xrightarrow{p} \lambda$ (say). Hence, the OLS estimator using all weekday data does not converges to either θ or θ_M where $\theta \neq \theta_M$ and the TTH does not hold.

Second, the dynamic analysis may result in a misleading one because of the above false estimation. Note that a moving average form of (1) is written as

$$P_t = (1 - L\theta_t)^{-1}\delta_t u_t$$

and an impulse response analysis (IRS) is conducted by the lag polynomial $(1 - L\theta_t)^{-1}\delta_t$ where L is a time lag operator. However, if the TTH (that is a misspecification if $\theta \neq \theta_M$) is assumed, then the IRS is falsely conducted by the lag polynomial $[1 - L(\theta\lambda + \theta_M(1-\lambda))]^{-1}$.

Third, in a unit root process of P_t assuming the TTH, a standardized shock u_t constantly affects to the future period's P_{t+k} as $\partial P_{t+k} / \partial u_t = 1$ for all $k > 0$. However, if the TTH is violated as $|\theta_W| < 1$, then $\lim_{k\to\infty} \partial P_{t+k} / \partial u_t = 0$. It is because we may write

$$P_t = \theta\theta_W^2 P_{t-3} + u_{Mt} \quad \text{if} \quad t \in D_M; \tag{8}$$

and thus the effect of shock at time t to P_{t+k} is reduced for every future weekend by the $\theta_W^2 (< 1)$. ■

It is interesting that time discounting may explain the 'weekend effect'; Monday stock returns are lower than those of the other weekdays. To observe this, note that the conditional expectation of the Monday return from the last Friday is given as:

$$E_{t-3}(P_t - P_{t-3}) = (\theta\theta_W^2 - 1)P_{t-3} < 0 \quad \text{if} \quad t \in D_M$$

from equation (8) if $|\theta_W| < 1$.

In the following section, we propose two test statistics of the TTH. The two are based upon the normalized differences of autoregressive coefficients and error variances between Monday and the other weekdays.

III. Limit Distributions of Test Statistics

We derive the formula for parameters θ_W and δ, and construct a test statistic from it directly. Those parameters may be compactly interpreted as the differences of *autoregressive coefficients* and *error variances* on Monday and other weekdays. In particular, we assert that the null hypothesis in (7) may equivalently be stated as

$$H_{01} : \theta_M = \theta \quad \text{and} \quad H_{02} : \sigma_M^2 = \sigma^2 . \tag{9}$$

To show this, note that the variance of u_{Mt} is written as:

$$\sigma_M^2 = \sigma^2(\theta_W^2\theta^2\delta^2 + \theta^2\delta^2 + 1) \geq \sigma^2 \tag{10}$$

from equation (6), where $\sigma_M^2 = Eu_{Mt}^2$. Thus, if the TTH is violated due to a nonzero δ, then heteroskedasticity arises in a daily asset price model. The violation of TTH may be a factor to explain the well-known heteroskedasticity of the daily frequency asset price model.[5]

Now note $\theta_W = 1$ if and only if $\theta_M = \theta$ because $\theta_W^2 = \theta_M / \theta$ where $0 < \theta$, $\theta_W \leq 1$ under Assumption 2.1. Further we may solve for δ^2 in Equation (10),

[5] The existence of weekend may be a reason for the heteroskedasticity; however, there could be other reasons such as crisis, volatility clusters, etc. The ARCH in Engle (1982) is a classic example reflecting these regularities.

which is a parameter representing the size of shock during weekends:

$$\delta^2 = \frac{\sigma_M^2 - \sigma^2}{\sigma^2\theta(\theta_M + \theta)}$$

from (5) and (6). Therefore $\delta = 0$ if and only if $\sigma_M^2 = \sigma^2$ because $\sigma^2\theta(\theta_M + \theta) \neq 0$ under Assumption 2.1.

To build the test statistics, we first estimate θ, θ_M, σ^2 and σ_M^2 through the following steps. First, the coefficient θ is estimated as

$$\hat{\theta} \equiv \frac{\sum_{t\in D_{TF}} P_{t-1}P_t}{\sum_{t\in D_{TF}} P_{t-1}^2}$$

from the OLS regression in Equation (3). Second, the error variance σ^2 is estimated from $\hat{\theta}$ as $\hat{\sigma}^2 \equiv (4n)^{-1}\sum_{t\in D_{TF}} \hat{u}_t^2$ where $\hat{u}_t = P_t - \hat{\theta}P_{t-1}$.

Third, the coefficient θ_M is estimated as

$$\hat{\theta}_M \equiv \frac{\sum_{t\in D_M} P_{t-3}P_t}{\sum_{t\in D_M} P_{t-3}^2}$$

by using the OLS regression in Equation (4). Finally, the error variance σ_M^2 is estimated from $\hat{\theta}_M$ as $\hat{\sigma}^2 \equiv n^{-1}\sum_{t\in D_M} \hat{u}_t^2$ where $\hat{u}_t = P_t - \hat{\theta}_M P_{t-3}$.

For the test of H_{01}, we introduce following statistic

$$T_{1n} = \frac{\hat{\theta}_M - \hat{\theta}}{\hat{\sigma}\sqrt{\frac{5}{\sum_{t\in D_{TF}} P_{t-1}^2}}}.$$

Let $B(r)$ denote a standard Brownian motion. Then, the T_{1n} statistic has the following limit distribution:

Theorem 3.1 *Suppose that Assumption 2.1 and* H_0 *hold*. (a) *Suppose* $|\theta| < 1$. *Then* $T_{1n} \xrightarrow{d} \mathbf{N}(0,1)$.

(b) *Suppose* $\theta = 1$, *then*[6]

[6] We may also derive the local-to-unity asymptotics of T_{1n} when μ is a nonrandom constant as $\theta = 1 + \mu/n$. Then, we may show that the Brownian motion is just replaced by a diffusion process. For instance, following Bobkoski (1983), note the processes $n^{-1/2}u_{[n.]} \rightarrow_d \sigma B_\mu(r)$, where B_μ is a diffusion process on the unit interval satisfying $dB_\mu(r) = \mu B_\mu(r) + dB(r)$ with $B_\mu(0) = 0$.

$$T_{1n} \xrightarrow{d} \frac{\dfrac{\int_0^1 B(r)dB(r)}{\int_0^1 B(r)^2 dr} - \dfrac{\int_0^1 B(r)dB(r)}{4\int_0^1 B(r)^2 dr}}{\sqrt{\dfrac{5}{16\int_0^1 B(r)^2 dr}}}. \tag{11}$$

All proofs are in appendix. Now another test statistic of the null hypothesis $H_{02}: \sigma_M^2 = \sigma^2$ is defined as

$$T_{2n} \equiv \frac{\hat{\sigma}_M^2 - \hat{\sigma}^2}{\sqrt{\frac{5}{4}[\hat{E}u_t^4 - (\hat{\sigma}^2)^2]}}, \tag{12}$$

where $\hat{E}u_t^4 = (4n)^{-1}\sum_{t\in D_{TF}} \hat{u}_t^4$.

The asymptotic distribution of T_{2n} test statistic under the null hypothesis is provided as:

Theorem 3.2 *Suppose that Assumption 2.1 holds and* $\hat{E}u_t^4 \xrightarrow{p} Eu_t^4 < \infty$.[7] *Then, under the null* H_0, *we get*

$$T_{2n} \xrightarrow{d} \mathbf{N}(0,1).$$

In the following section, we evaluate the size and power of the suggested tests through Monte Carlo simulations.

IV. Simulation and Applications Using S&P 500 Data

We consider the data generation process of P_t with a time varying coefficients as

$$P_t = \theta_t P_{t-1} + \delta_t u_t, \tag{13}$$

where $\theta_t = 1$ and $\delta_t = 1$ for $t = 7k + i$ and $i = 1,2,3,4,5$ (*representing weekdays*); and $\theta_t = \theta_W$ and $\delta_t = \delta$ for $t = 7k + i$ and $i = 6,7$ (*representing weekend*) where $k(=0,1,\ldots,n-1)$ denotes a sample number index. The P_0 is initially set to 0 in simulation.

First, we compute the critical values for T_{1n} test under $H_0: \theta_W = 1$ (or $\theta_M = \theta$) for different sample numbers: $n = 25$, 50, 100, 250, and 500.[8] This sample reflects

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[7] We may readily show that if (u_t) is an i.i.d normal process, then $\hat{E}u_t^4 \xrightarrow{p} Eu_t^4$ if $\theta = 1$.

[8] It is the same as in Dickey and Fuller (1981), which also computes critical values of statistics for

that the limit distribution of T_{1n} test statistic is a nonstandard one. Each simulation consists of 25,000 replications. The following Table 2 presents the results. Noteworthy is that the critical values were slightly wider than those of the standard normal distribution except when $n = 500$. For instance, the 95% confidence interval is [-1.975, 1.964] when $n = 100$.[9]

[Table 2] Critical values of T_{1n} test when $\theta = 1$; $\theta_W = 1$ and $\delta = 0$

Sample size	Probability that T_{1n} is less than entry							
n	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-2.362	-2.001	-1.651	-1.296	1.292	1.653	1.981	2.325
50	-2.395	-2.026	-1.700	-1.306	1.297	1.652	1.986	2.352
100	-2.358	-1.975	-1.647	-1.296	1.287	1.652	1.964	2.327
250	-2.335	-1.986	-1.647	-1.276	1.291	1.663	1.997	2.394
500	-2.319	-1.948	-1.657	-1.295	1.273	1.643	1.947	2.312

Then we further compute the rejection rates (power) for the T_{1n} test after setting the nominal size at the 5% for the hypotheses

$$H_1 : \theta_W = 0.1, \ldots, 0.9, 1$$

letting $n = 100$, for each $\delta = 0, .1, \ldots, .3$. Table 3 below shows the results. We may see that the rejection rates increase as θ_W deviates from 1 ($H_{01} : \theta_W = 1$). However they do not quite depend on the size of δ representing the variance of the error term.

[Table 3] Rejection rates of T_{1n} test for 5% size when n=100

	δ			
θ_w	0	0.1	0.2	0.3
.9	0.162	0.144	0.150	0.146
.8	0.259	0.220	0.231	0.172
.7	0.312	0.335	0.359	0.209
.6	0.393	0.425	0.247	0.363
.5	0.557	0.386	0.378	0.330
.4	0.449	0.443	0.504	0.494
.3	0.528	0.370	0.365	0.368
.2	0.419	0.432	0.588	0.343
.1	0.401	0.437	0.424	0.422

the unit root test, which is asymptotically a function of the Brownian motion.

[9] We suspect the estimators' difference form of term $\hat{\theta}_M - \hat{\theta}$ may crowd out the asymmetry that typically appears in a unit root test statistic.

Second, we also compute the rejection rates for T_{2n} test for the nominal size 5% where the sample size n is 100. Next Table 4 shows the results. The size of T_{2n} test is computed as 0.052 when $\delta = 0$ and $\theta_W = 1$, which is very close to the nominal size of 5%. Note that the rejection rate increases sharply as δ deviates from 0: it is close to 1 for the values $\delta \geq 0.7$. Hence, the power of T_{2n} test is good conditional on $\theta_W = 1$. However the size of the T_{2n} test seems to be distorted as θ_W falls below 1. Thus, we recommend a comparison of the test results of T_{1n} test (that is checking the null $H_{01} : \theta_W = 1$) in order to prevent any potential size distortion.

[Table 4] Rejection rates of T_{2n} test for 5% size when n=100

	θ_w			
δ	1	0.9	0.7	0.5
0	0.052	0.078	0.146	0.220
.1	0.056	0.083	0.150	0.227
.2	0.101	0.124	0.184	0.248
.3	0.234	0.245	0.282	0.328
.4	0.491	0.492	0.488	0.494
.5	0.778	0.761	0.727	0.697
.6	0.943	0.931	0.902	0.870
.7	0.991	0.988	0.976	0.961
.8	0.999	0.999	0.999	0.991
.9	1.00	0.999	0.999	0.999
1	1.00	1.00	0.999	0.999

Finally, we apply the tests to the daily S&P 500 data of the United States for the years 2000～2011. Table 5 depicts the descriptive statistics of the data. The median for the Monday return (log difference of indices) is the smallest one among the other weekday returns. Further, the standard deviation for the Monday return is the largest one among the stock returns that confirms the inequality (10). The empirical cumulative distribution functions of stock returns in Graph 1 also confirmed that the Monday return (DMON) has a wider range than the other weekdays (DTUES, DWED, DTHUR, DFRI).

Table 6 shows the test results. For the whole period data, the T_{1n} test did not reject the null $H_{01} : \theta_M = \theta$ at the 5% level where the alternative is $H_{A1} : \theta_M \neq \theta$. However the T_{2n} test rejected the null $H_{02} : \sigma_M^2 = \sigma^2$ at the 5% level where the alternative is $H_{A2} : \sigma_M^2 \neq \sigma^2$. Those jointly mean that the TTH is partially rejected: shocks might occur during weekends while the time discounting does not occur during weekends.[10]

[10] However we do not know whether two days of a weekend are too short for the statistically significant time discounting estimation or not.

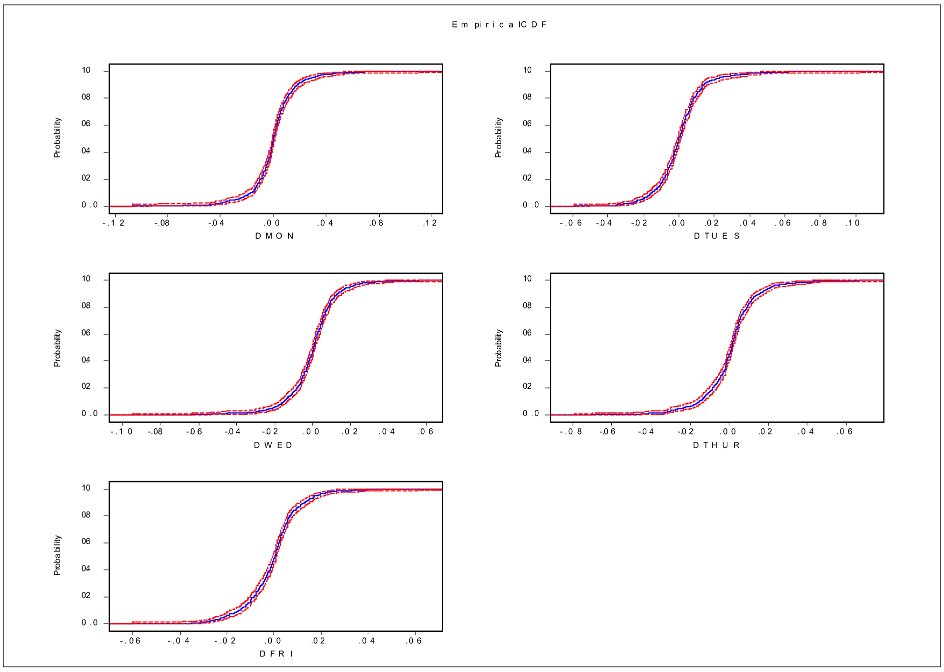
[Table 5] Total descriptive statistics for S&P 500 stock return

	dMon[1)]	dTue[2)]	dWed[2)]	dThu[2)]	dFri[2)]
Mean	0.0000	0.0004	-0.0005	0.00030	-0.0006
Median	0.0000	0.0001	0.0004	0.0012	0.0003
Maximum	0.1095	0.1024	0.0557	0.0669	0.0613
Minimum	-0.1064	-0.0591	-0.0946	-0.0792	-0.0600
Std. Dev	0.0184	0.0142	0.0136	0.0143	0.0118
Observations	515	516	516	516	516

1) dMon=Monday(current week)-Friday(last week)

2) dAday=Aday(current week)-A-1day(current week)

[Graph 1] Empirical cumulative distribution functions for S&P500 stock return



Tests for each individual year were also conducted. For every year between 2000 and 2011, the null H_{01} was not rejected at the 10% level as to whether θ is 1 (and critical values in Table 2 are used) or not (and standard critical values are used). In addition, for the years 2003 and 2009, the null hypothesis H_{02} was rejected at the 1% level. It implies that statistically significant weekend shocks arise during these years. Note that the invasion of Iraq by coalition forces occurs in 2003. In addition, the global financial crisis in the years 2008 and 2009 was triggered in the U.S. banking system. Clearly, wars find no breaks during weekends and governments can announce intervention policies any day of the week, including the weekends, during economic crises.

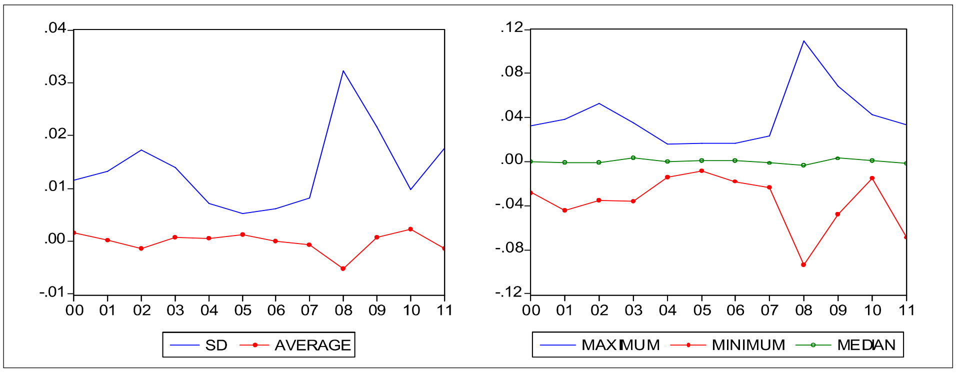
[Table 6] Yearly estimation and test results for S&P 500 index

Year	coefficients $\hat{\theta}$	$\hat{\theta}_M$	test statistics[1)] T_{1n}	T_{2n}	sample numbers[2)] n_1	n_2
2000	0.999	1.000	1.279	-1.303	44	45
2001	0.999	1.000	0.215	0.217	42	46
2002	0.999	0.999	-0.432	0.342	43	47
2003	1.000	1.000	0.075	4.268***	43	46
2004	1.000	1.000	0.396	-0.206	43	44
2005	0.999	1.000	1.067	-1.642	44	45
2006	1.000	1.000	-0.554	-0.084	43	45
2007	1.000	0.999	-0.588	-1.493	43	47
2008	0.999	0.999	-0.504	1.528	43	46
2009	1.000	1.000	-0.439	2.938***	43	45
2010	0.999	1.000	1.315	-0.818	44	44
2011	1.000	0.999	-0.683	1.200	41	42
total(2000～2011)	0.999	0.999	0.013	2.164**	516	542

1) *, **, and *** denote significance at the 10%, 5%, and 1% levels, respectively.
2) n_1 denotes the number of days used in T_{1n}, and n_2 denotes the number of days used in T_{2n}.

Sample statistics of Friday-Monday returns of each year in the following Graph 2 show that their volatility measured by the max-minimum and standard deviation (SD) are relatively large during the ears of invasion of Iraq by coalition forces and the global financial crisis. These statistics partly confirm the above test results because the weekends during these years generated larger shocks when compared with other normal years.

[Graph 2] Sample statistics of Friday-Monday returns of each year



For the robustness check of our estimation results, we estimated the following equation in Connolly (1989: p136):

$$R_t = \beta_0 + \beta_1 M_t + \varepsilon_t \tag{14}$$

where $R_t \equiv P_t - P_{t-1}$ is the market return index (log-differential of indices) and M_t is the dummy variable identifying Monday, respectively. A negative value of coefficient β_1 may imply the 'weekend effect'; Monday stock returns are lower than those of the other weekdays. Table 7 reports the estimates of (14). Note the estimates of β_0 and β_1 are not significant at the 5% level, which signifies that there is not the weekend effect (or Monday stock returns *are not* lower than those of the other weekdays) after 2000. Chow, Hsiao and Solt (2003, p432), and Connolly's (1989; Table 2) results also indicate a weakening of the weekend effect in the post-1975 era. This result is similar to our T_{1n} test results in Table 6; the *autoregressive coefficients* on Monday and other weekdays are not different with each other.

[Table 7] Estimates of Model (14) for 2000～2011[1)]

Year	coefficients β_0	coefficients β_1	F-test value	P-value
2000	-0.001	0.002	1.213	0.271
	(-0.867)	(1.101)		
2001	-0.000	-0.000	0.035	0.851
	(-0.387)	(-0.187)		
2002	-0.001	-0.001	0.207	0.649
	(-0.741)	(-0.455)		
2003	0.000	0.000	0.026	0.871
	(1.008)	(0.162)		
2004	0.000	0.000	0.057	0.811
	(0.620)	(0.239)		
2005	0.000	0.001	1.605	0.206
	(-0.202)	(1.267)		
2006	0.000	-0.000	0.172	0.678
	(1.195)	(-0.414)		
2007	0.000	-0.001	0.384	0.535
	(0.473)	(-0.620)		
2008	-0.001	-0.002	0.357	0.550
	(-0.770)	(-0.597)		
2009	0.001	-0.001	0.055	0.813
	(0.698)	(-0.236)		
2010	0.000	0.002	1.678	0.196
	(-0.029)	(-1.295)		
2011	0.000	-0.001	0.380	0.537
	(0.175)	(-0.617)		
total(2000～2011)	-0.000	-0.000	0.011	0.915
	(-0.139)	(-0.106)		

1) t-statistic is reported in parentheses below the coefficient estimates. Under the F-test value, there is the F-statistic for the hypothesis $\beta_1 = 0$ and its p-value is in the next column.

This approach to estimate equation (14) has two differences with our method. First, it tries to test the return shift for Monday while our method tries to test the shift of autoregressive coefficient of Monday. Second, the volatility increase during the weekend is not considered while our method checks it through the T_{2n} test. These differences come from the different purposes of tests on the weekend effect; our test is directly trying to test the TTH while the test using equation (14) is to check the abnormality of returns between the market close on Fridays and the market close on Mondays.

V. Conclusion

Standard daily financial time series analyses using autoregressive (AR) models typically disregard weekends following the trading time hypothesis (TTH) because the relevant assets of the models are not traded (and thus, their prices are not observed) on weekends. However, weekends may affect asset prices through time discounting as well as through shocks/news occurring on weekends. In this regard, we suggest a test for the TTH by using an AR(1) model, where many asset prices are closely approximated by an AR(1) process. The proposing test statistics are based upon the differences of AR coefficients and error variances between Monday and the other weekdays. Asymptotic normality of the suggested test statistics under the TTH and model stationarity is proved. Under the model of nonstationarity, the test statistic is asymptotically pivotal/non-standard; moreover, the critical values are given from the Monte Carlo simulations. In an application for the United States S&P 500 data during the years 2000～2011, we found that the TTH was rejected, particularly during the years of war and financial crisis. We also confirmed a weakening of the weekend effect as depicted in Chow, Hsiao and Solt (2003), and Connolly's (1989) results. The results require us to revise the dynamic analyses using a time series model of asset prices considering the weekends

Finally, this study will be extended by including more general AR(p) or VAR(p) models based on the solid results for the AR(1) model of this paper.

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Appendix A: Proofs

Before proceeding, note that the null TTH implies that (P_t) has the time indices as follows.

[Table 8] Sample indices without weekends

	Mon	Tues	Wed	Thur	Fri	Mon	Tues	···	Mon	Tues	Wed	Thur	Fri
t	1	2	3	4	5	6	7	···	5n-4	5n-3	5n-2	5n-1	5n

Hence, D_M and D_T each has n sample number and D_{TF} has $4n$ sample number. The following proofs are based upon these sample numbers. For the derivation of standard asymptotics, we will later use the following summation equalities,

$$\begin{aligned}\textstyle\sum_{t\in D_{TF}} P_{t-1}u_t &= \textstyle\sum_{t\in D_T} (P_{t-1}u_t + P_t u_{t+1} + P_{t+1}u_{t+2} + P_{t+2}u_{t+2}) \\ &= \textstyle\sum_{t\in D_M} (P_t u_{t+1} + P_{t+1}u_{t+2} + P_{t+2}u_{t+3} + P_{t+3}u_{t+4}) \end{aligned} \tag{15}$$

and similarly

$$\begin{aligned}\textstyle\sum_{t\in D_{TF}} \hat{u}_t^2 &= \textstyle\sum_{t\in D_T} (\hat{u}_t^2 + \hat{u}_{t+1}^2 + \hat{u}_{t+2}^2 + \hat{u}_{t+3}^2) \\ &= \textstyle\sum_{t\in D_M} (\hat{u}_{t+1}^2 + \hat{u}_{t+2}^2 + \hat{u}_{t+3}^2 + \hat{u}_{t+4}^2)\ . \end{aligned} \tag{16}$$

Now we derive the limit distributions of T_{1n} and T_{2n} statistics under the TTH.

Theorem 3.1 First, we write

$$\begin{aligned} & n^{1/2}(\hat{\theta}_M - \hat{\theta}) \qquad (17) \\ &= n^{1/2}(\hat{\theta}_M - \theta_M) - n^{1/2}(\hat{\theta} - \theta) \\ &= \frac{n^{-1/2}\sum_{t\in D_M} P_{t-1}u_t}{n^{-1}\sum_{t\in D_M} P_{t-1}^2} - \frac{n^{-1/2}\sum_{t\in D_{TF}} P_{t-1}u_t}{n^{-1}\sum_{t\in D_{TF}} P_{t-1}^2} \\ &= \left(\frac{1}{n^{-1}\sum_{t\in D_M} P_{t-1}^2}, \frac{-1}{n^{-1}\sum_{t\in D_{TF}} P_{t-1}^2} \right) \times \\ & \quad n^{-1/2} \begin{pmatrix} \sum_{t\in D_M} P_{t-1}u_t \\ \sum_{t\in D_{TF}} P_{t-1}u_t \end{pmatrix} \\ &= \left(\frac{1}{n^{-1}\sum_{t\in D_M} P_{t-1}^2}, \frac{-1}{n^{-1}\sum_{t\in D_{TF}} P_{t-1}^2} \right) \times \end{aligned}$$

$$n^{-1/2}\sum_{t\in D_M}\begin{pmatrix} P_{t-1}u_t \\ P_t u_{t+1}+P_{t+1}u_{t+2}+P_{t+2}u_{t+3}+P_{t+3}u_{t+4}\end{pmatrix}$$

from the definition because $\theta_M=\theta$ under the TTH for the second equality; the last equality comes from Equality (15).

Now we derive the limit distribution of the last term in (17).

(a) The first case is $|\theta|<1$. From the stationarity and TTH, we get

$$n^{-1}\sum_{t\in D_M}P_{t-1}^2\xrightarrow{p}\Delta\equiv\frac{\sigma^2}{1-\theta^2}\quad\text{and}\quad n^{-1}\sum_{t\in D_{TF}}P_{t-1}^2\xrightarrow{p}4\Delta,$$

and thus

$$=\left(\frac{1}{n^{-1}\sum_{t\in D_M}P_{t-1}^2},\ \frac{-1}{n^{-1}\sum_{t\in D_{TF}}P_{t-1}^2}\right)\xrightarrow{p}\frac{1}{\Delta}\left(1,\ -\frac{1}{4}\right)$$

and

$$n^{-1/2}\sum_{t\in D_M}\begin{pmatrix} P_{t-1}u_t \\ P_t u_{t+1}+P_{t+1}u_{t+2}+P_{t+2}u_{t+3}+P_{t+3}u_{t+4}\end{pmatrix}\xrightarrow{d}\mathbf{N}\left(0,\sigma^2\begin{bmatrix}\Delta & 0\\ 0 & 4\Delta\end{bmatrix}\right)\quad(18)$$

using the central limit theorem for a martingale difference sequence where $E[(P_{t-1}u_t)(P_{s-1}u_s)]=0$ for zero asymptotic covariances because (u_t) is an i.i.d process, and for $t\neq s$.

Therefore we get

$$n^{1/2}(\hat{\theta}_M-\hat{\theta})\xrightarrow{d}\mathbf{N}\left(0,\ \sigma^2\frac{5}{4\Delta}\right)$$

from (17) and (18). Finally, we get

$$\frac{n^{1/2}(\hat{\theta}_M-\hat{\theta})}{\sigma\sqrt{\frac{5}{4\Delta}}}\xrightarrow{d}\mathbf{N}(0,\ 1)$$

and

$$T_{1n}=\frac{\hat{\theta}_M-\hat{\theta}}{\hat{\sigma}\sqrt{\frac{5}{\sum_{t\in D_{TF}}P_{t-1+i}^2}}}=\frac{n^{1/2}(\hat{\theta}_M-\hat{\theta})}{\hat{\sigma}\sqrt{\frac{5}{n^{-1}\sum_{t\in D_{TF}}P_{t-1+i}^2}}}\xrightarrow{d}\mathbf{N}(0,\ 1)$$

using Slutsky's theorem, because $\hat{\sigma}\xrightarrow{p}\sigma$ and $n^{-1}\sum_{t\in D_{TF}}P_{t-1}^2\xrightarrow{p}4\Delta$.

(b) The second case is when $\theta=1$. Note we multiply n to the denominator and the numerator;

$$T_{1n}=\frac{n(\hat{\theta}_M-\hat{\theta})}{\hat{\sigma}\sqrt{\frac{5}{n^{-2}\sum_{t\in D_{TF}}P_{t-1}^2}}}.$$

Then we show that the numerator converges in the distribution to

$$\begin{aligned}
n(\hat{\theta}_M - \hat{\theta}) &= \frac{n^{-1}\sum_{t\in D_M} P_{t-1}u_t}{n^{-2}\sum_{t\in D_M} P_{t-1}^2} - \frac{n^{-1}\sum_{t\in D_{TF}} P_{t-1}u_t}{n^{-2}\sum_{t\in D_{TF}} P_{t-1}^2} \\
&= \frac{n^{-1}\sum_{t\in D_M} P_{t-1}u_t}{n^{-2}\sum_{t\in D_M} P_{t-1}^2} - \frac{\frac{4n}{n}(4n)^{-1}\sum_{t\in D_{TF}} P_{t-1}u_t}{\frac{(4n)^2}{n^2}(4n)^{-2}\sum_{t\in D_{TF}} P_{t-1}^2} \\
&\xrightarrow{d} \frac{\int_0^1 B(r)dB(r)}{\int_0^1 B(r)^2 dr} - \frac{\int_0^1 B(r)dB(r)}{4\int_0^1 B(r)^2 dr}
\end{aligned}$$

and

$$\begin{aligned}
\hat{\sigma}\sqrt{\frac{5}{n^{-2}\sum_{t\in D_{TF}} P_{t-1}^2}} &= \hat{\sigma}\sqrt{\frac{5}{\frac{(4n)^2}{n^2}(4n)^{-2}\sum_{t\in D_{TF}} P_{t-1}^2}} \qquad (20) \\
&\xrightarrow{d} \sigma\sqrt{\frac{5}{\sigma^2 16\int_0^1 B(r)^2 dr}} = \sqrt{\frac{5}{16\int_0^1 B(r)^2 dr}}
\end{aligned}$$

from the functional central limit theorem and continuous mapping theorem with $\hat{\sigma} \xrightarrow{p} \sigma$.

Consequently, the statistic T_{1n} converges in the distribution as

$$T_{1n} \xrightarrow{d} \frac{\dfrac{\int_0^1 B(r)dB(r)}{\int_0^1 B(r)^2 dr} - \dfrac{\int_0^1 B(r)dB(r)}{4\int_0^1 B(r)^2 dr}}{\sqrt{\dfrac{5}{16\int_0^1 B(r)^2 dr}}}$$

from (19) and (20).

Theorem 3.2 Note that we may write

$$\begin{aligned}
n^{1/2}(\hat{\sigma}_M^2 - \hat{\sigma}^2) &= n^{1/2}[(\hat{\sigma}_M^2 - \sigma_M^2) - (\hat{\sigma}^2 - \sigma^2)] \qquad (21) \\
&= n^{1/2}[n^{-1}\sum_{t\in D_M}(\hat{u}_t^2 - \sigma^2) - (4n)^{-1}\sum_{t\in D_{TF}}(\hat{u}_t^2 - \sigma^2)] \\
&= n^{-1/2}\sum_{t\in D_M}(\hat{u}_t^2 - \sigma^2) - \frac{1}{4}n^{-1/2}\sum_{t\in D_{TF}}(\hat{u}_t^2 - \sigma^2) \\
&= (1, -1/4)\begin{pmatrix} n^{-1/2}\sum_{t\in D_M}(\hat{u}_t^2 - \sigma^2) \\ n^{-1/2}\sum_{t\in D_{TF}}(\hat{u}_t^2 - \sigma^2) \end{pmatrix} \\
&= (1, -1/4)n^{-1/2}\sum_{t\in D_M}\begin{pmatrix} \hat{u}_t^2 - \sigma^2 \\ \hat{u}_{t+1}^2 + \hat{u}_{t+2}^2 + \hat{u}_{t+3}^2 + \hat{u}_{t+4}^2 - 4\sigma^2 \end{pmatrix}
\end{aligned}$$

because $\sigma_M^2 = \sigma^2$ for the second equality under the TTH; the last equality comes from Equality (16).

Now we show that

$$n^{-1/2} \sum_{t\in D_M} \begin{pmatrix} \hat{u}_t^2 - \sigma^2 \\ \hat{u}_{t+1}^2 + \hat{u}_{t+2}^2 + \hat{u}_{t+3}^2 + \hat{u}_{t+4}^2 - 4\sigma^2 \end{pmatrix} \tag{22}$$
$$= n^{-1/2} \sum_{t\in D_M} \begin{pmatrix} u_t^2 - \sigma^2 \\ u_{t+1}^2 + u_{t+2}^2 + u_{t+3}^2 + u_{t+4}^2 - 4\sigma^2 \end{pmatrix} + o_p(1)$$

because we may derive

$$\begin{aligned} & n^{-1/2} \sum_{t\in D_M} \hat{u}_{t+i}^2 \\ &= n^{-1/2} \sum_{t\in D_M} (P_{t+i} - \hat{\theta} P_{t-1+i})^2 \\ &= n^{-1/2} \sum_{t\in D_M} u_{t+i}^2 + (\theta - \hat{\theta})^2 n^{-1/2} \sum_{t\in D_M} P_{t-1+i}^2 + 2(\theta - \hat{\theta}) n^{-1/2} \sum_{t\in D_M} P_{t-1+i} u_{t+i} \\ &= n^{-1/2} \sum_{t\in D_M} u_{t+i}^2 + o_p(1) \end{aligned}$$

for $i = 0,1,2,3$ and 4 from (i) and (ii):

(i) If $|\theta| < 1$, then

$$(\theta - \hat{\theta})^2 n^{-1/2} \sum_{t\in D_M} P_{t-1+i}^2 = [n^{1/2}(\theta - \hat{\theta})]^2 n^{-3/2} \sum_{t\in D_M} P_{t-1+i}^2 \xrightarrow{p} 0$$

and

$$2(\theta - \hat{\theta}) n^{-1/2} \Sigma_{t\in D_M} P_{t-1+i} u_{t+i} \xrightarrow{p} 0$$

because $n^{1/2}(\theta - \hat{\theta})$, $n^{-1/2}\Sigma_{t\in D_M} P_{t-1+i}u_{t+i}$ and $n^{-1} \sum_{t\in D_M} P_{t-1+i}^2$ are $O_p(1)$.

(ii) If $\theta = 1$, then

$$(\theta - \hat{\theta})^2 n^{-1/2} \sum_{t\in D_M} P_{t-1+i}^2 = [n(\theta - \hat{\theta})]^2 n^{-5/2} \sum_{t\in D_M} P_{t-1+i}^2 \xrightarrow{p} 0$$

and

$$2(\theta - \hat{\theta}) n^{-1/2} \Sigma_{t\in D_M} P_{t-1+i} u_{t+i} = 2n^{1/2} (\theta - \hat{\theta}) n^{-1} \Sigma_{t\in D_M} P_{t-1+i} u_{t+i} \xrightarrow{p} 0$$

because $n(\theta - \hat{\theta})$, $n^{-1}\Sigma_{t\in D_M} P_{t-1+i}u_{t+i}$ and $n^{-2} \sum_{t\in D_M} P_{t-1+i}^2$ are $O_p(1)$.

Further note that

$$n^{-1/2} \sum_{t\in D_M} \begin{pmatrix} u_t^2 - \sigma^2 \\ u_{t+1}^2 + u_{t+2}^2 + u_{t+3}^2 + u_{t+4}^2 - 4\sigma^2 \end{pmatrix} \xrightarrow{d}$$
$$\mathbf{N}\left(0,\ E(u_t^2 - \sigma^2)^2 \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\right) \tag{23}$$

from the central limit theorem where $E(u_{t+i}^2 - \sigma^2) = 0$, $E(u_{t+i}^2 - \sigma^2)^2 = Eu_{t+i}^4 - (\sigma^2)^2$ and $E(u_{t+j}^2 - \sigma^2)(u_{t+k}^2 - \sigma^2) = 0$ for $j \neq k = 0,1,2,3$ and because $(u_t^2 - \sigma^2)$ is a mean-zero i.i.d. process from the assumption.

Therefore we get

$$n^{1/2}(\hat{\sigma}_M^2 - \hat{\sigma}^2) \xrightarrow{d} N\left(0, \frac{5}{4}[Eu_t^4 - (\sigma^2)^2]\right)$$

from (21), (22) and (23) jointly. Finally, we prove the claimed result as

$$T_{2n} \equiv \frac{\hat{\sigma}_M^2 - \hat{\sigma}^2}{\sqrt{\frac{5}{4}[\hat{E}u_t^4 - (\hat{\sigma}^2)^2]}} \xrightarrow{d} \mathbf{N}(0,1)$$

because $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$ and $\hat{E}u_t^4 \xrightarrow{p} Eu_t^4$ from the assumption and using Slutsky's theorem.