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THE TERM STRUCTURE OF INTEREST RATES, THE STOCHASTIC BOND PRICE, AND IMMUNIZATION

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I. INTRODUCTION

Over the past several years, a number of studies have been published evaluating bond portfolio immunization strategies based on the concept of duration.[1] Immunization is defined as obtaining a realized yield over a planning period which is greater than, or equal to the promised yield to maturity. As defined originally by Macaulay,[2] duration is a weighted average of the payment periods where the weights are related to the present value of the payments in each period. As long as the duration of a bond portfolio is equal to the planning period, risky assets are effectively converted into a riskless asset with a known yield for any given holding period.

In the case of default- and option-free bonds, realized yields on coupon bonds over any holding period may differ from the yields to maturity at the same time of purchase, either because the bonds are sold before maturity or because the coupon payments are not fully reinvested to maturity at a yield equal to the promised yield at the time the coupon bonds were purchases. If the yield realized is less than the yield expected (or promised) at the time of purchase, the expected terminal wealth is not realized.

The duration of a coupon bond is always shorter than the term to maturity. A duration based immunization investment strategy implies selling a coupon bond before maturity. If interest rates change after the bond is purchased, the investor is subject to two risks: (1) a price risk when the bond is sold before maturity, (2) a coupon reinvestment risk resulting from reinvesting the coupons at interest rates different from the yield to maturity on the bond at the time of purchase. The final effect of these risks on realized returns varies in opposite directions and depends on changes in interest rates. If there are increases in interest rates, the market value

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[1]Fisher and Weil (1971), Bierwag (1977, 1978, 1979), Kaufman (1978), Bierwag, Kaufman, and Khang (1978), Bierwag and Khang (1979), Cox, Ingersoll, and Ross (1979), Khang (1979, 1983), Marshall and Yawitz (1982), Bierwag, Kaufman, and Toevs (1983), Brennan and Schwartz (1983) and Fong and Vasicek (1984).

[2]Macaulay, Frederick R (1938), see references.

of a bond will decrease but the return from reinvestment of the coupon payments will increase, and vice versa.

The immunization of a bond from changes in interest rates after purchase requires that the price risk and coupon reinvestment risk offset each other. Therefore, immunization duration must be the time period at which the price risk and the coupon reinvestment risk of a bond (or bond portfolio) are opposite in direction but equal in magnitude. Even though a duration strategy seems to provide immunization for a portfolio of bonds, the definition of duration that achieves immunization is dependent upon the nature of the random shocks that are assumed to affect interest rates after the purchase of the portfolio.

In general, conventional immunization based on duration contains two ad hoc assumptions. First, it assumes that there is only one state variable (or one factor) which defines the term structure of interest rates, i.e., all interest rates are perfectly correlated. Second, it assumes that the sensitivity of the term structure to shifts in this state variable is unity for all maturities. There is, however, no reason to expect a priori that all interest rates are perfectly correlated or that sensitivity is unity for all maturities.

Duration strategy analysis does not ask whether the stochastic behavior of the term structure implied by the state variable is consistent with equilibrium. Duration theory is most reasonably interpreted as a theory of the covariance matrix of bond returns which rests upon the implicit assumption of a single underlying source of uncertainty. While duration theories have tended to ignore the equilibrium implications of their restrictions on the covariance matrix, modern theories of bond pricing explicitly use the equilibrium conditions in arriving at their estimates of the covariance structure.[3]

In recent papers, Cox, Ingersoll, and Ross (1979), Brennan and Schwartz (1983), and others[4] have investigated an immunization condition when interest rates are generated by a continuous stochastic process consistent with an equilibrium condition, using either a single state variable model or two state variable model. Depending on the postulation of the interest rate process, there is a measure like duration such that the portfolio is immunized if a proper value of this measure is met. This assumes a continuous reorganizing of the portfolio.

In this paper, we will relax the assumptions made in duration theory and postulate a three state variable model of bond pricing in the context of immunization in equilibrium and extend to an n state variable model of immunization. This paper consists of four main sections. Section Two deals with background, assumptions, and notations of the model. Section Three proposes and discusses the immunization model. Section Four extends the immunization model to an n state

[3]Estimation of the covariance structure is not the primary objective of these theories.

[4]Vasicek (1977), Richard (1978), and Dothan (1978).

variable model and Section Five discusses some issues relevant to empirical estimations.

II. BACKGROUND, ASSUMPTIONS, AND NOTATION

1. Background

According to modern theories of the term structure of interest rates, the value of a default-free bond of any given maturity can be represented as a deterministic function of n state variables, each of which is assumed to follow a continuous stochastic diffusion process. Using arbitrage arguments, we can derive a single partial equilibrium condition of a bond price for immunization which implies that, in equilibrium, a partial differential equation should be satisfied by the values of all such bonds. The differential equation involves the same number of unknown parameters as state variables which reflect the market valuation of the risk in connection with the stochastic state variables.

Several studies[5] have been based on either single or two state variable models. In this study, we use a three state variable model.

Traditionally, the theory of the term structure has been cast in terms of the relationship between the forward rates which are inherent in the term structure and the corresponding expected future spot rates of interest. Thus, the typical version of the expectation hypothesis postulates that forward rates are equal to expected future spot rates.[6] In contrast to the expectations hypothesis, the liquidity premium hypothesis asserts that forward rates are always greater than the corresponding expected future spot rates by a liquidity premium sufficient to compensate investors for the assumed greater capital risk inherent in longer-term bonds. The market segmentation hypothesis can be regarded as a modification of the liquidity premium hypothesis to accommodate a positive or negative liquidity premium on long-term bonds. This hypothesis assumes that long-term bonds are not necessarily more risky than short-term bonds for investors who have long-term planning horizons, so that the values of bonds of different maturities are determined by the preferences of investors with different planning horizons. Therefore, in the market segmentation hypothesis, forward rates may bear no systematic relationship to expected future spot rates.

Now, in case of a single state variable model,[7] it is assumed that, since the instantaneous interest rate follows a Markov process, all that is known about future

[5]See footnote 4 and Brennan and Schwartz (1979).

[6]It is argued that this assumption is incompatible with universal risk neutrality. See Cox, Ingersoll, and Ross (1985a).

[7]Merton (1971), Vasicek (1977), and Dothan (1978).

rates is impounded in the current instantaneous interest rate. The price of a default free bond of any maturity may be represented as a function of current instantaneous interest rates and time. In a single state variable (factor) model, price changes in bonds of all maturities are perfectly correlated. Such a model also implies that bond prices do not depend on the path followed by the spot rate in reaching its current level. This means that the whole term structure of interest rates may be explained using only the current instantaneous interest rate, regardless of any deterministic shifts in preferences over time. Clearly, this does not seem to correspond to reality.

Brennan and Schwartz (1979, 1983) argue that they take a step toward a more realistic approach to the pricing of bonds with different maturities by allowing changes in the instantaneous interest rate to depend on the long-term rate of interest as well as on the current instantaneous interest rate, so that the long-term rate and the instantaneous rate follow a joint Gauss-Markov process. Their expansion of the state vaiable from one rate of interest to two is, it is contended, to reflect the assumption that the current long-term rate of interest contains information about future values of the spot rate of interest, as is postulated in both the expectations and the liquidity premium hypotheses.

Brennan and Schwartz take the long-term rate of interest as exogenous and try to explain the intermediate portion of the yield curve in terms of its two extreme state variables. Taking the two extreme values and attempting to explain the intermediate portion of the term structure of interest rates might lead to misspecification problems or errors in the coefficient estimates of bond pricing. In other words, the two extreme state variable model may not account for substantially all the variation in the term structure.

Therefore we assume that the intermediate term interest rate together with the instantaneous and long term rates of interest determine the term structure, based on the modified liquidity premium and market segmentation hypothesis. We assume that intermediate forward rates always exceed the corresponding expected future spot rates by a liquidity premium and that there is also an intermediate-term bond market so that the prices of different maturities are determined by the preferences of investors with different planning periods. The model developed here, viewed simply as a model of the term structure, may be less ambitious than the single state variable model or complicated multifactor model as in Cox, Ingersoll, and Ross (1985a). However, it is anticipated that the major contribution of the present model will be an alternative bond immunization strategy with manageability in the practical sense, including option risk and default risk which are not dealt with in conventional immunization theory based on duration analysis.[8]

[8]To incorporate default risk, it might be necessary to have the full information about the stochastic process of the factors which affect the capital markets.

2. Assumptions

The major assumptions of the model are as follows:

(A.1). Let r, i, and l denote the spot (instantaneous) rate, the intermediate-term rate, and long-term rate of interest, respectively. These rates are assumed to follow a joint stochastic, continuous Markov process.

(A.1.1). There are no jumps in the state variables, i.e., no large instantaneous shocks by these variables.

(A.1.2). Changes in each interest rate are partially interdependent; both the mean and variance of the change in each interest rate depend on the value of the other interest rates as well as its own value.

(A.2). The price of a default-free discount bond at time t of maturity T, promising $ 1 at maturity, is assumed to be a function of the current values of the interest rates, r, i, and l, time, t, and maturity, T.

(A.3). The market is efficient. That is, there are no transaction costs, no taxes, information is available to all investors simultaneously, and every investor acts rationally (prefers more wealth to less, and uses all available information).

(A.4). Investors attempt to maximize expected returns, for a given level of risk exposure, over a known and certain planning period.

3. Notation

From assumption (A.1), we can express changes in each interest rate as follows:

$$
\begin{aligned}
(1)\quad dr &= a_1(r,i,l,t,T)dt + b_1(r,i,l,t,T)dZ_1 \\
di &= a_2(r,i,l,t,T)dt + b_2(r,i,l,t,T)dZ_2 \\
dl &= a_3(r,i,l,t,T)dt + b_3(r,i,l,t,T)dZ_3
\end{aligned}
$$

where t and T represent calendar time and maturity, respectively, $t \leqslant T$, and dZ_1, $i=1, 2, 3$, are Wiener processes with $E(dZ_i)=0$, $dZ_i^2=dZ_j^2=dt$, $dZ_idZ_j=\sigma dt$, where $i,j=1, 2, 3$, $i \neq j$; $a_1(\bullet)$, $a_2(\bullet)$, and $a_3(\bullet)$ are the expected instantaneous rates of change in the spot, intermediate-term, and long-term rate of interest, respectively, and $b_1^2(\bullet)$, $b_2^2(\bullet)$, and $b_3^2(\bullet)$ are instantaneous variances in the rate of change in the three interest rates. σ is the instantaneous correlation between the unexpected changes in the two interest rates.

The justification for assumption (A.1.2) is that the expected change in the spot rate of interest will depend on the intermediate-term and the long-term rate of interest, as long as the intermediate-term and the long-term rate carry information about future values of the spot rate. Similar justifications can be applied to the cases of the intermediate-term and long-term rate.

Equation (1) also implies that the unanticipated changes in the three interest rates are correlated. If it is assumed that the change in the instantaneous rate is due to a change in expectations of the instantaneous rate of inflation, the same interpretation can be made for the intermediate and long rate case.

The instantaneous covariance among interest rates, $i,j = r, i, l$, is

$b_{ri}\,dt = E(dr)(di)$, $b_{il}\,dt = E(di)(dl)$, and $b_{rl}\,dt = E(dr)(dl)$

For notational convenience we denote the instantaneous covariance matrix by

$$(2) \qquad \Sigma = \begin{bmatrix} b_r^2 & b_{ri} & b_{rl} \\ b_{ir} & b_i^2 & b_{il} \\ b_{lr} & b_{li} & b_l^2 \end{bmatrix}$$

which is assumed to be positive. Also for notational simplicity, denote the vector of stochastic returns by

$$(3) \qquad bdZ = \begin{bmatrix} b_r dZ_r \\ b_i dZ_i \\ b_l dZ_l \end{bmatrix}$$

We also denote the price of a default-free bond as postulated in assumption (A.2) by

$$(4) \quad P(T) = P(r,i.l,t,T)$$

III. THE MODEL

1. Stochastic Bond Prices

It follows from equation (1), (2) and (4), by applying Ito's lemma,[9] that the stochastic process for the price of a discount bond (zero coupon bond) is

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[9]Suppose we have a number of stochastic processes describable by $dp_i/p_i = a_i\,dt + b_i\,dZ_i$, $i = 1, \ldots, n$, and σ_{ij} as the correlation coefficient between the Wiener processes dZ_i and dZ_j. Then let $F(p_1, \ldots, p_n, t)$ be a function which is at least twice differentiable, which obviously, depends on the stochastic processes. Ito's lemma gives the rule for finding the differential of $Y = F(p_1, \ldots p_n, t)$. Specifically,

$$dY = \sum_{i=1}^{n} (\delta F/\delta p_i)\,dp_i + (\delta F/\delta t)\,dt$$
$$+ (1/2) \sum_{i=1}^{n} \sum_{j=1}^{n} (\delta^2 F/\delta p_i \delta p_j)\,dp_i\,dp_j,$$

is the stochastic differential of the function $F(\bullet)$. The product $dp_i dp_j$ is defined by $dZ_i dZ_j = \sigma_{ij}\,dt$, $i,j = 1, \ldots, n$, and $dZ_i dt = 0$, $i = 1, \ldots, n$.

$$(5)\quad dP/P = \mu(r,i,l,t,T)dt + n_1(r,i,l,t,T)dZ_1 + n_2(r,i,l,t,T)dZ_2 + n_3(r,i,l,t,T)dZ_3$$

where
$$\mu(r,i,l,t,T) = [P_1a_1 + p_2a_2 + P_3a_3 + P_4 + (1/2)P_{11}b_1^2 + (1/2)P_{22}b_2^2 + (1/2)P_{33}b_3^2 + P_{12}\sigma_{12}b_1b_2 + P_{23}\sigma_{23}b_2b_3 + P_{13}\sigma_{13}b_1b_3]/P$$

$$n_1(r,i,l,t,T) = P_1b_1/P$$
$$n_2(r,i,l,t,T) = P_2b_2/P$$
$$n_3(r,i,l,t,T) = P_3b_3/P$$

and $P_1 = \delta P/\delta r$, $P_2 = \delta P/\delta i$, $P_3 = \delta P/\delta l$, $P_4 = \delta P/\delta t$, etc.

2. The Zero-Risk and Equilibrium Conditions

An arbitrage argument does not rely upon any particular risk attitude of investors as maintained by Cox and Ross (1976). Hence, in finding solutions to the resulting equation for bond prices, it is expedient to assume the simplest risk attitude possible which is risk-neutrality. The problem may be solved for all investors by assuming all expected instantaneous rates of return on market-traded securities are equal. If we arbitrarily choose four bonds with distinct maturities T_1, T_2, T_3, and T_4, and combine them into a portfolio with proportion k_1 of maturity T_1, k_2 of T_2, k_3 of T_3, and k_4 of T_4, we can derive the equilibrium relationship between expected returns on bonds of different maturities. The rate of return on this portfolio, denoted by dF/F, is given by

$$\begin{aligned}(6)\quad dF/F = {} & [k_1\mu(T_1) + k_2\mu(T_2) + k_3\mu(T_3) + k_4\mu(T_4)]dt \\ & + [k_1n_1(T_1) + k_2n_1(T_2) + k_3n_1(T_3) + k_4n_1(T_4)]dZ_1 \\ & + [k_1n_2(T_1) + k_2n_2(T_2) + k_3n_2(T_3) + k_4n_2(T_4)]dZ_2 \\ & + [k_1n_3(T_1) + k_2n_3(T_2) + k_3n_3(T_3) + k_4n_3(T_4)]dZ_3\end{aligned}$$

The rate of return on this portfolio will be immunized (in a non-stochastic world) if we choose k_1, k_2, k_3 and k_4 so that the coefficients of dZ_1, dZ_2, and dZ_3 in equation (6) are zero. That is, in the absence of transaction costs, this portfolio may be continuously revised to be immunized so that at each time t the portfolio has no instantaneous variance. Therefore,

$$\begin{aligned}(7)\quad & k_1n_1(T_1) + k_2n_1(T_2) + k_3n_1(T_3) + k_4n_1(T_4) = 0 \\ & k_1n_2(T_1) + k_2n_2(T_2) + k_3n_2(T_3) + k_4n_2(T_4) = 0 \\ & k_1n_3(T_1) + k_2n_3(T_2) + k_3n_3(T_3) + k_4n_3(T_4) = 0\end{aligned}$$

In order to avoid arbitrage possibilities, the rate of return on this portfolio should be equal to the instantaneous riskless rate of interest in equilibrium. Hence,

$$(8)\quad k_1\mu(T_1) + k_2\mu(T_2) + k_3\mu(T_3) + k_4\mu(T_4) = r$$

The zero risk conditions given in the system of equations (7), and the no arbitrage condition (8), imply a linear risk-return relationship for discount bonds in the four portfolio proportions. Equations (7) and (8) provide a solution for arbitrary T_1, T_2, T_3, and T_4 if and only if for all T there exist functions $\theta_1(r,i,l,t)$, $\theta_2(r,i,l,t)$, and $\theta_3(r,i,l,t)$, independent of maturity such that

$$(9)\quad \mu(T) - r = \theta_1(\bullet)n_1(T) + \theta_2(\bullet)n_2(T) + \theta_3(\bullet)n_3(T)$$

The term, $\mu(T) - r$, in equation (9) represents the instantaneous excess expected rate of return on a bond of maturity T. Equation (9) obviously implies an equilibrium relationship which restricts the relative risk premium on a portfolio of bonds of different maturities. Therefore, the excess expected rate of return (the instantaneous risk premium) on a bond of maturity T can be denoted as the sum of three components. θ_1 can be intrepreted as the market prices of the spot interest rate risk, since $n_1(T)$ is the instantaneous standard deviation of the return on the bond of maturity T induced by unexpected changes in the spot rate of interest. Similarly θ_2, and θ_3 are the market prices of the intermediate-term, and the long-term rate of interest risk, respectively. These market prices are determined in equilibrium as a function of investors' preferences, endowments, and productive opportunities.

3. The Solution for Bond Price and Immunization

A partial differential equation for the price of a discount bond can be derived from (9). If we substitute the definitions for $\mu(\bullet)$, n_1, n_2, and n_3 into (9) and rearrange terms, we get

$$\begin{aligned}(10)\quad & (1/2)P_{11}b_1^2 + (1/2)P_{22}b_2^2 + (1/2)P_{33}b_3^2 + P_{12}\sigma_{12}b_1b_2 \\ & + P_{23}\sigma_{23}b_{23} + P_{13}\sigma_{13}b_1b_3 + P_1(a_1 - \theta_1b_1) \\ & + P_2(a_2 - \theta_2b_2) + P_3(a_3 - \theta_3b_3) + P_4 - rP = 0\end{aligned}$$

If we define X as a vector of adjusted risk premiums, then

$$(11)\quad X = \begin{bmatrix} \theta_1b_1 \\ \theta_2b_2 \\ \theta_3b_3 \end{bmatrix}$$

We obtain a solution for discount bond prices from the partial differential equation (10), using equation (2), (3), and (11), together with the boundary condition, $P(r,i,l,T,T) = 1$ and $P>0$.

The unique general solution to (10) is

$$(12)\ P(r,i,l,t,T) = E_t \exp \left[- \int_t^T r(v)dv - \int_t^T (1/2)\, X' \Sigma^{-1} X\, dv - \int_t^T X' \Sigma^{-1}\, bdZ \right]$$

where E_t is the expectation operator conditional upon the state variables at time t. The third integral is defined as a stochastic integral, which is a stochastic process with zero mean for all $t \leq T$. (see Friedman (1964) and Arnold (1974) for more details).[10] Therefore, the price of a discount bond at time t of maturity T depends upon parameters of the stochastic processes for r, i, and l as well as the underlying functions $\theta_1(\bullet)$, $\theta_2(\bullet)$, and $\theta_3(\bullet)$. Generally, θ_i, $i = 1, 2, 3$, is negative if investors value liquidity and expected returns on long-term bonds are greater than on short-term bonds. Similarly, θ_i, $i = 1, 2, 3$, is positive if short-term bonds are considered riskier. The prices of regular coupon bonds can be derived by treating each coupon as a discount bond so that a single coupon bond is a portfolio of discount bonds.

Once we obtain bond price formula (12), it is possible for investors to be immunized from interest rate risk by the zero risk condition, equation (7), and the no arbitrage condition, equation (8), if and only if we know the parameters of the stochastic process for the state variables as well as the estimates of the risk premium for each state variable. All we have to do is to select the portfolio of bonds for immunization which satisfy the simultaneous equation systems (7) and (8). Then, the value of the asset portfolio, at each instant, will be precisely equal to that of the liabilities. Under the conditions we have assumed of continuous trading, diffusion processes, and frictionless markets, we know that perfect immunization is feasible.

IV. THE GENERALIZATION

1. Assumptions and Notation

In general, we can apply the immunization strategy to the n state variable case. The model is general in that it allows the price of a discount bond to depend on an arbitrary number of stochastic state variables (factors) following an arbitrary Ito process. We can incorporate into this immunization strategy factors such that option-risk and default-risk, which might affect the price of a discount bond, with appropriate specifications for these factors.

[10] See Appendix for proof.

Suppose the price at time t of a discount bond promising \$ 1 on maturity T is assumed to be a function of n state variables, $s_1, s_2, \ldots, s_n$, $t \leq T$.

$$(13)\quad P(t,T) = P(t,T,s_1 s_2, \ldots, s_n)$$

The state variables are assumed to follow a multivariate joint stochastic diffusion process:

$$(14)\quad ds_i = a_i(t,T,s_i \ldots)dt + b_i(t,T,s_i \ldots)dZ_i,\ i=1, \ldots, n$$

where $a_i(\bullet)$ is the drift component of the state variable, b_i^2 is the instantaneous variance rate of the state variable s_i, and the dZ_i is a standard Wiener process.

$$(15)\quad E(dZ_i\ dZ_j) = \begin{bmatrix} dt\quad, i = j \\ \sigma_{ij}dt,\ i \neq j \end{bmatrix}$$

or,

$$E(ds_i\ ds_j) = \begin{bmatrix} b_i^2 dt,\ i = j \\ b_{ij}dt,\ i \neq j \end{bmatrix}$$

We also assume that the riskless rate can be expressed as a function of the same set (or subset) of stochastic state variables, $r = r(t,s_i, \ldots)$.

2. The General Bond Pricing and Immunization Conditions

The bond pricing model we are developing is very general, since the functional form of r, a_i, and b_i is arbitrary, and almost any reasonable continuous specification is possible. To develop an explicit bond pricing model, i.e., for perfect immunization, the functional form of r, a_i, and b_i must be theoretically or empirically specified.[11]

Assume that P(t,T,s) is continuous in t and s, with continuous partial derivatives with respect to t and s, and with continuous second partial derivatives with respect to s. Applying Ito's lemma, the stochastic process for the price of a discount bond is

$$(16)\quad dP/P = \mu(\bullet)dt + \sum_{i=1}^{n} n_i dZ_i$$

$$\text{where } \mu(\bullet) = \{\sum_{i=1}^{n} P_i b_i + \sum_{i=1}^{n}\sum_{j=1}^{n} [(1/2)P_{ij}b_{ij}] + P_t\}/P$$

$$n_i(\bullet) = (P_i b_i)/P$$

[11]See Cox, Ingersoll, and Ross (1985a) for one of the more elegant theories concerning the functional relation of the instantaneous spot rate to underlying economic stochastic factors.

P_i denotes for partial derivative with respect to s_i, b_{ij} for covariance between ds_i and ds_j (if $i=j$, $b_{ij} = b_i^2$). Thus, the rate of return on a portfolio with a proportion k_j of maturity T_j is

$$(17)\quad dF/F = \sum_{j=1}^{n+1} k_j\mu(T_j)dt + \sum_{i=1}^{n}\sum_{j=1}^{n+1} k_j n_i(T_j)dZ_i$$

$$\text{where } \sum_{j=1}^{n+1} k_j = 1$$

Assuming no short sales, we can always choose a portfolio with a proportion k_j of T_j such that the portfolio has no risk for immunization,

$$(18)\quad \sum_{i=1}^{n}\sum_{j=1}^{n+1} k_j n_i(T_j) = 0$$

If capital markets are sufficiently perfect (i.e., no taxes, no transaction costs in portfolio revision), then market equilibrium requires that the rate of return on the portfolio should be equal to the instantaneous riskless interest rate r. Therefore,

$$(19)\quad \sum_{j=1}^{n+1} k_j\mu(T_j) = r$$

Equations (18) and (19) will have a unique solution, provided that there exist functions (or estimates) of θ_i such that

$$(20)\quad \mu(T) - r = \sum_{i=1}^{n} \theta_i(t,s)n_i(T)$$

Equation (20) implies that the instantaneous excess expected rate of return on a bond of maturity T can be expressed as a linear combination of the adjusted risk premium (or $\theta_i n_i$).

Now, if an underlying stochastic state variable is tradable, $\theta_i = (a_i - r)/b_i$. When s_i is not tradable, θ_i must be empirically estimated or theoretically specified.[12]

[12]See Ross (1976), Cox et al. (1985a) for details. The market price of risk θ_i, $i=0, 1, \ldots, n$, may depend on time and all of the underlying stochastic state variables s. In general, it is expected that θ depends on factors such as the level of aggregate wealth and the level of investor risk premium. If all the stochastic state variables affecting bond prices are tradable, then we could make the assumption of risk neutrality to simplify the solution technique. However, the state variables affecting the bond price are not always tradable.

3. The General Solution for Bond Prices and Immunization

An equilibrium bond pricing equation for any pure discount bond P(t,T,s) paying \$ 1 at maturity T is obtained by substituting for μ and n_i, $i=1, \ldots, n$ from (16) into (20). After rearranging,

$$(21)\ (1/2) \sum_{i=1}^{n} \sum_{j=1}^{n} P_{ij} b_{ij} + \sum_{i=1}^{n} (a_i - \theta_i b_i) P_i + P_t - rP = 0$$

With boundary condition, $P(T,T,s) = 1$, the solution to (21) is as follows:

$$(22)\ P(t,T,s) = Et\{\exp[A(T)]\}$$

$$A(T) = [-\int_t^T r(v)dv - \int_t^T (1/2) X \Sigma^{-1} X dv - \int_t^T X' \Sigma^{-1} bdZ]$$

where

$$X = \begin{bmatrix} \theta_1 b_1 \\ \theta_2 b_2 \\ \cdot \\ \cdot \\ \theta_n b_n \end{bmatrix}$$

$$bdZ = \begin{bmatrix} b_1 dZ_1 \\ b_2 dZ_2 \\ \cdot \\ \cdot \\ b_n dZ_n \end{bmatrix}$$

Σ^{-1} = inverse of the $(n \times n)$ covariance matrix Σ, where Σ is assumed to be full rank.

With a general bond price formula, (22), perfect immunization would be achieved if we choose a portfolio with a proportion k_j of maturity T_j such that equations (18) and (19) are satisfied simultaneously. For that purpose, it is assumed that we have full information about all the relevant parameters, as well as the appropriate estimates of risk premiums which satisfy equation (20) for all the state variables. Depending on the knowledge we have about market structure, perfect immunization may or may not be realized. Under the condition of full knowledge of the stochastic process for the relevant state variables, we can realize perfect immunization with no interest rate risk and, then, at every moment of time, the value of the asset portfolio must be equal to that of the liability portfolio. In general, if there are k factors, the immunizing portfolio must contain $k+1$ bonds.

V. EMPIRICAL APPLICATIONS

In applying the model for empirical estimation, there are several, relevant issues.

1. Alternative Stochastic Processes

Some may suggest an alternative stochastic process as follows,

$$(23)\ ds(t,s) = [e(t) + f(t)s(t)]dt + b(t,s)dZ(t)$$

The first order linear process in equation (23) has been termed as an Ornstein-Uhlenbeck process, or a normal backwardation process. Assuming that $f<0$, future short-term rates will asymptotically tend toward their long-term normal level of $-e/f$. If $f=0$, then the normal backwardation process degenerates to the random walk process such as (14) which has no normal level, and when $f>0$, the process becomes explosive.

While the normal backwardation process and random walk process are consistent with the data,[13] both models have different theoretical implications. The random walk model implies that the spot rate, for example, may drift off to infinite positive and negative values. As shown by Merton (1973), the possibility of an infinite negative yield implies that extremely long-term discount bonds tend to infinite positive values. The normal backwardation process does not have this problem, provided that $f<0$. At the same time, the normal backwardation process does not permit the transient occurrence of negative short-term rates. Since it is assumed that a negative spot rate will never occur in reality, the normal backwardation process is an inappropriate process when the short-term rate is close to zero. Moreover, the restriction that short-term rate approach the long-term rate also requires further justification.[14]

Another point to be made is that the pricing formula for a discount bond in equation (22) is not generally valid. If we do not know the probability distribution of the exponent, A(T), the expectation in (22) cannot be evaluated. In this case we can use numerical methods directly on the general form of partial differential equation (21). Even if we do know the probability distribution of A(T), it might be necessary to use numerical analysis to evaluate the expectation in (22).

The solution to equation (22) depends on the specific underlying stochastic pro-

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[13]A different statistical test on different data may prove these processes to have merit. These are empirical issues.

[14]While the long-term rate may proxy a normal long-term level for the short-term rate, there is no justification to ensure equal coefficients on the past long-term and short-term rates.

cess. Cox, Ingersoll, and Ross (1985a) solve (22) for the case of a univariate square root process (i.e., $b = g\sqrt{s}$, where g is deterministic). Dothan (1978) solves (22) for the case of the univariate geometric random walk process (i.e., $b = gs$). Finally, Vasicek (1977) solves (22) for the case of the normal backwardation process (i.e., $b = g$).

All these arguments relate to empirical issues and the model could be modified for empirical estimation depending on the market process assumption used. In setting up an immunization strategy, one of the important points to be considered in the practical sense is the problem of cost efficiency for that strategy. The application of a particular strategy require the estimation of the underlying state variables' stochastic processes, which might be bothersome or subject to error. If the strategy also requires complicated numerical methods to compute bond risk measures, such efforts may not be justified when the objective is to immunize a portfolio containing simple instruments.

2. Risk Premium

Brennan and Schwartz (1979), in their two state variable model, derive the market price for long-term interest rate risk. They assume the existence of a consol bond with a constant risk premium. Based on this constant long-term risk premium, they estimate the short-term interest rate risk by empirical methods. Therefore, they could avoid the need for deriving a risk premium function by taking, as one of the state variables, the long-term rate of interest which is inversely proportional to an asset price, the price of the consol bond. They then argue that the risk associated with that factor can be hedged away. Their use of the consol, while theoretically useful, is unrealistic in the practical sense and may explain why their empirical work shows the possibility of misspecifications or errors in their estimation.[15]

In a contemporary paper, Cox, Ingersoll, and Ross (1985a) have also constructed models of bond pricing whith incorporate two state variables. The advantage of their models lies in the endogeneity of the long-term rate of interest, but it is obtained by introducing two risk premium functions into the practical differential equation for bond prices, which significantly complicates the issues of empirical estimation. They avoid the estimation problems imposed by the two risk premium functions by the preferences of the representative investor explicitly.[16] This might, however, lead to an aggregation problem. Under explicit preference for the

[15]As the internal rate on a long-term bond is used as a proxy for the consol rate, market price of risk for long-term bond estimated via a coupon bond is not the same as that estimated via a consol bond. Furthermore, long-term government bonds are typically callable, and the impact of this provision will induce some bias in periods of time where probability of future call is positive.

[16]They set up a general equilibrium model for a representative individual.

representative investor, the conditions for aggregation are quite restrictive. It might impose some restriction on the family of utility functions that can be used, given that it is necessary to estimate the parameters characterizing the utility function.

In the current model, we simply assume that the market price for the risk of each state variable is constant and that it can be estimated empirically. In his 1976 paper, Ross shows that, if security returns are generated by a linear factor model, then, under quite general conditions, the equilibrium excess rate of return of any security can be represented as linear combination of the factor risk premiums. The risk premium of the j^{th} factor is defined as the excess expected rate of return on a security, or portfolio, which has only the risk of the j^{th} factor. Although our underlying model is much more fully developed, the coefficients, θ_i, are the factor risk premia in the Ross sense. In that sense, however, we should add a further assumption to our model, as made by Ross; that is, the level of wealth does not affect the factor risk premium.

VI. CONCLUSION

In this paper, we relaxed the assumption made in duration theory and postulated a three state variable model of bond pricing in the context of immunization in equilibrium and further extended to include an n state variable model of immunization. So far, no attempts have been made to develop the concept of immunization using a bond pricing model. Thus, it is anticipated that the major contribution of the model developed here will be an alternative bond immunization strategy with manageability in the practical sense, including option-risk and default-risk, which are not dealt with in conventional immunization theory based on duration analysis.

Based on the modern theories of the term structure of interest rates, we know that the value of a default-free bond of any given maturity can be represented as a deterministic function of n state variables, each of which is assumed to follow a continuous stochastic diffusion process. Using arbitrage arguments, immunization can be shown to require a single partial equilibrium condition on bond prices, which implies that, in equilibrium, a partial differential equation should be satisfied by the values of all bonds in the protfolio. The differential equation involves the same number of unknown parameters as state variables which reflect the market valuation of the risk in connection with the stochastic state variables.

Once we derive the stochastic bond price formula, it is possible for investors to be immunized from interest rate risk by the zero risk condition and the no arbitrage cndition in the model, if and only if, we know the parameters of the stochastic process for the state variables as well as the estimates of the risk premium for each state variable. All we have to do is to select the portfolio of bonds for immunization which satisfies the simultaneous equation system. Then, the value of the asset portfolio, at each instance, will be precisely equal to that of the liabilities. That is, under the assumptions of continuous trading, an explicit diffusion process and an efficient markets, perfect immunization is possible. Therefore, realized rate of return of bond portfolios will be greater than or equal to the rate of return expected at the initial investment.

For immunization purpose, it is required that we should have full information about all the relevant parameters for all the state variables. That is, application of the model to the immunization strategy needs further specification of the macroeconomic structure of the state variables. To make the things simple, application of the model requires (1) estimation of the coefficients of the state variables' stochastic process, (2) estimation (or assumption) of the market prices of risk, and (3) identification of the economic state variables related to the term structure of interest rates.

Depending on the knowledge we have about market structure, perfect immunization may or may not be achieved. For more reliable results, further study is necessary on both the theoretical points and the empirical estimation in this field.

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APPENDIX

Following a similar method with Vasicek (1977) and Richard (1978), if we define

$$\text{(A1)}\quad h(u) = P(r,i,l,u,T)\, A(u)$$

where

$$\text{(A2)}\quad A(u) = \exp\left[-\int_t^u r(v)\,dv - \int_t^u (1/2)\, X'\,\Sigma^{-1}\, X\,dv - \int_t^u X'\,\Sigma^{-1}\, b\,dZ\right],$$

then, by applying Ito's differential rule to the process h(u),

$$\text{(A3)}\quad dh(u) = AdP + PdA + (1/2)\,P\,(dA)^2 + (dP)\,(dA)$$
$$= A(u)\left[\,(1/2)\sum_{i=1}^{3}\sum_{j=1}^{3} P_{ij}b_{ij} + \sum_{i=1}^{3}(a_i - \theta_i b_i)P_i + P_t - rP\right]dv + A(u)[P_rP_iP_l) - P\,X'\,\Sigma^{-1}]bdZ$$

By virtue of equation (10)

$$\text{(A4)}\quad dh(u) = A(u)[(P_r\ P_i\ P_l) - P\,X'\,\Sigma_{-1}]bdZ$$

If we take expectations

$$\text{(A5)}\quad E_t[dh(u)] = 0$$

Therefore, h(u) is a martingale,

(A6) $E_t[h(u)|h(t)] = h(t)$

If we evaluate (A6) at $u = T$, then we get (12) by boundary condition, $P(r,i,l,T,T) = 1$ and $A(t) = 1$.