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ON THE INDETERMINACY AND ROBUSTNESS OF FINANCIAL EQUILIBRIA IN ECONOMIES WITH RESTRICTIONS ON PORTFOLIO CHOICE

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I. INTRODUCTION

There has been intensive research on economies with "incomplete markets" under uncertainty and hence almost everything is exposed to us about the existence, optimality and the structure of equilibria.[1] When we consider an economy with a sequence of markets under uncertainty, what distinguishes an incomplete market structure from a complete one is whether it is "essentially sequential"[2] or not. For instance, the economy with a complete set of Arrow's contingent securities is a well known case with a sequence of markets which are not essentially sequential and hence are complete.

There can be many different sources that make an economy essentially sequential. A typical one is that there is insufficient number of assets available in the markets as instruments for transferring agent's wealth across time and states. Once the markets are incomplete, the qualitative aspect of the economy depends upon the property of assets, i.e., whether they are "real" or "financial" assets in the sense defined in the current literature.[3] Cass[4], Duffie[9] and Werner[15] examined the existence of a competitive equilibrium with incomplete financial markets. Balasko and Cass[3], Cass[5] and Geanakoplos and Mas-Colell[11] examined the number and the structure of equilibrium allocations. Geanakoplos and Polemarchakis[12] and Younes[16] examined the optimality property of equilibrium allocations. Also there is a bunch of other works extending the arguments to production economies with incomplete markets.

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[1]See the bulk of the literature about "Incomplete Markets" since Radner's seminal paper(1972), which is well documented in [3].

[2]For instance, see Hahn, "On the Notion of Equilibrium in Equilibrium in Economics"(1974) for the early discussion about the equilibrium concept in economies with a sequence of markets.

[3]This distinction is based upon whether returns are paid off in terms of units of account or in terms of commodity bundles. But, it seems to be more appropriate to distinguish assets with "endogeneous" returns from those with "exogeneous" returns.

On the other hand, a certain kind of restrctions on portfolio holdings can make an economy essentially sequential even with sufficient number of assets. Then, the economy may reveal similar properties as in the canonical economy with incomplete markets. For instance, consider an economy with a complete set of contingent securities, but with some restrictions on portfolio holdings. Since a contingent security is not a financial asset, but also can be interpreted as a real asset which delivers one unit of numeraire commodity in the corresponding state,[4] it may be interesting to examine the effect of those restrictions on the existence of an equilibrium and on the set of equilibrium allocations. There is no obvious economic explanation why people want to hold their portfolios only from a restricted subset instead of the original set of portfolios. It may be the case that there are some people who want to hold their portfolios only as combinations of some "mutual funds" formed from contingent securities. Furthermore, there are certain reasons to consider an economy with these features.

First, it is the case in the current literature on incomplete markets that real securities yield only nominal indeterminacy in the space of prices and financial securities yield real indeterminacy of equilibria in the space of allocations on the other hand. But, this dichotomy seems to be somewhat misleading because contingent securities can be interpreted in both ways and there can be some kind of real indeterminacy with a complete set of contingent securities and with some restrictions on their choices.

Second, since the theory of incomplete markets can be viewed as the theory of "rationing", as was pointed out by Younes[16], we can do more exercises in this line by imposing many different types of restrictions on portfolio choices.

Third, by choosing a certain type of restrictions on portfolio choices, we can not only reproduce exactly the same conclusions as those from the canonical model of incomplete financial markets, but aoso extend the argument to a broader class of economic situations.

Finally, if we consider an ecoinomy with non-contingent securities and define a "restricted participation" in financial markets such that different consumers may hold their portfolios from different subsets of financial assets, then any type of restricted participations in financial markets can be completely characterized by the appropriate choice of restrictions on the portfolios of contingent securities choices.

Here, people are assumed to hold their portfolios as some combinations of mutual funds formed from contingent securities and different people may want to hold different mutual funds. Then, the optimal choices of individuals and the equilibria in this economy will be examined through the analysis of the role of contingent securities prices. Also, the flexibility in choosing such restrictions will allow us to take a close look at the "robustness" of a "complete market" hypothesis

[4]This has been pointed out by many people, for instance, by J. Dreze etc..

in an economy with a sequence of markets.

II. THE MODEL

1. Overall Description of the Model

A pure exchange economy is considered in the simple context, encompassing today($t=0$) and tomorrow($t=1$). There are a finite number of consumers indexed by $h \in H=\{h : h=1,..,m\}$ and a finite number of commodities in each period, indexed by $c \in C=\{c : c=1,..,L\}$. There are uncertain states of nature tomorrow, summarized by a finite set $S=\{s : s=1,..,N\}$. Each consumer is endowed with a strictly positive vector of commodities, denoted by $e_h=(e_h(0), e_h(1),\ldots, e_h(N)) \in R_{++}^{(N+1)L}$ and hence $e=(e_1,..,e_m) \in (R_{++}^{(N+1)L})^m=\Omega$. His preference under uncertainty is represented by a utility function U_h:$X_h \rightarrow R$ such that

(u1) U_h is at least twice continuously differentiable.

(u2) U_h is strictly increasing.

(u3) U_h is strictly quasi-concave.

(u4) the closure of indifference surfaces are contained in $R_{++}^{(N+1)L}$) where $X_h=R_+^{(N+1)L}$ is a closed, convex consumption set.

As instruments to facilitate consumers' intertemporal allocations of consumption goods under uncertainty, there is a complete set of contingent securities. A state "s" contingent security costs $\pi(s)$ units of account (dollars) in $t=0$ and pays off exactly 1 unit of account only if state "s" occurs.

2. Financial Opportunities with Restrictions on Portfolio Choices

Unlike in the original paper by Arrow[1], there is no outside money and short sales of securities are allowed here. Furthermore, each consumer is assumed to hold his portfolio only from subset of R^N in the following sense.

Definition 1

A mutual fund "f" is a security composed of any fixed linear combination of the original contingent securities with weights adding up to one.

Thus, a mutual fund f can be represented by a vector $k^f=(k^f(1),\ldots,k^f(N))$ with $\sum_s k^f(s)=1$. There can be at most N economically different mutual funds formed from contingent securities.

Suppose that a consumer h is restricted to hold his portfolio only as combinations of some mutual funds, say, as combination of mutual funds i and j. Then, his portfolio $a_h=a_h(1),..,a_h(N))$ is denoted by $a_h=\alpha_h^i k^i+\alpha_h^j k^j$ for α^i, $\alpha^j \in R$. Thus, the set of his portfolio choices "A_h" is nothing but a subspace of R^N spanned by k^i and k^j. So, if he is supposed to hold his portfolio as combinations of M different mutual funds, A_h is an M-dim subspace of R^N spanned by $k^1,..,k^M$.

Thus, financial opportunities are represented by the collection of A_h, $\{A_h\}$ with possibly different linear constraints for different consumers.

Remark 1

There is no loss of generality in modelling exchanges of contingent securities as if they were exchanged separately at the given price vector $\pi=(\pi(1),\ldots,\pi(N))$ as long as the linear constraints on each consumer's portfolio choices are maintained. This means that it is not necessary to introduce the prices of mutual funds explicitly since the corresponding prices of mutual funds can be obtained always from the prices of contingent securities through the straightforward relation between them.[5]

3. Consumer's Problem and the Role of Cointingent Securities Prices

Each consumer will choose(x_h, a_h) as an optimal solution to the following maximization problem:

$$\begin{aligned}
&\text{Maximize } U_h\,(x_h) \\
&\text{s.t. } p(o)\,(x_h(o) - e_h(o)) \leqslant -\sum_s \pi(s) && (\mu_h(o)) \\
&\qquad p(s)\,(x_h(s) - e_h(s)) \leqslant a_h(s) && (\mu_h(s)) \text{ for } s=1,\ldots,N \\
&\text{and } x_h \in X_h,\ a_h \in A_h
\end{aligned}$$

Now, let's take a look at consumer's problem in detail.
The subscript "h" will be dropped for the notational convenience. Suppose that the constrained set of portfolio "A" is an M-dim subspace of R^N with $1 \leqslant M \leqslant N$. Then, we can define an (N-M)-dim orthogonal complement oif A in the following way:

$$O(A)=\{v \in R^N : v^T a = 0 \text{ for all } a \in A\}$$

with the following (N-M) linearly independent vectors as a basis of O(A) without loss of generality:

$$[v]=[v_1,\ldots,v_{N-M}]=\begin{bmatrix} v_1(1),\ldots,v_{N-M}(1) \\ \vdots \qquad \vdots \\ v_1(M),\ldots,v_{N-M}(M) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ I_{N-M} \end{bmatrix}$$

[5]There is a well defined mathematical relation between prices of mutual funds and prices of contingent securities. The famous Minkowski-Farkas lemma summarizes that relation with a slight modification. Suppose that NxM matrix [K] denotes the collection of M different mutual funds. Then, $q^T=\pi^T[K]$ where $\pi \in R^N_{++}$ and $q \in R^M_+$ are contingent securities and mutual funds prices respectively without arbitrage possibilities.

where I_{N-M} is an $(N-M)\times(N-M)$ identity matrix.

Since $v^T a=0$ for all $a \in A$, any $a=(a(1),.,a(N))$ must satisfy the following linear restrictions:

$$(1)\ a(M+1) = -\sum_{s=1}^{M} v_1(s)\, a(s),\ \ldots\ldots,\ a(N) = -\sum_{s=1}^{M} v_{N-M}(s)\, a(s)$$

After plugging these restrictions into the budget constraints, we can derive the firstorder conditions of consumer's maximization:

(2) $\partial U(x)/\partial x^c(o) = \mu(o)\, p^c(o)$

$\partial U(x)/\partial x^c(s) = \mu(s)\, p^c(s)$ for $c=1,\ldots,L$ and $s=1,\ldots,N$

(3) $\mu(o)\pi(1)-\mu(1) = (\mu(o)\pi(M+1)-\mu(M+1))\, v_1(1) + \ldots + (\mu(o)\pi(N)-\mu(N))\, v_{N-M}(1)$

$\vdots \qquad\qquad \vdots$

$\mu(o)\pi(M)-\mu(M) = (\mu(o)-(M+1)-\mu(M+1))\ v_1\ (M) + \ldots + (\mu(o)\pi(N)-\mu(N))v_{N-M}(M)$

and $(N+1)$ budget constraints with equalities.

Now, define $\delta=(\delta(1),\ldots,\delta(N))=(\mu(1)/\mu(0),\ldots,\mu(N)/\mu(0))$.

Then, we'll obtain the following relation from (3) after some manipulation:

$$(4)\ (\delta-\pi) = [v]\ (\delta-\pi)_{N-M}$$

where $(\delta-\pi)=(\delta(1)-\pi(1),\ldots,\delta(N)-\pi(N))$ and $(\delta-\pi)_{N-M}=(\delta(M+1)-\pi\ (M+1),\ldots,\ \delta(N)-\pi(N))$. Obviously $(\delta-\pi)\in O(A)$, implying that δ is an element of a translate of $O(A)$, i.e., $\delta \in T\ (A:\pi) \equiv O(A)+\{\pi\}$ for some given $\pi \varepsilon R^N_{++}$.

Since δ must be a strictly positive vector at the optimal solution, we can define the following $O(A:\pi)$ as the set of admissible δ:

$$O(A:\pi) = \{\delta \in R^N_{++}\ :\ \delta = v+\pi \text{ for } v \in O(A) \text{ and some } \pi \in R^N_{++}\ \}$$

Now, let $(x(p,\pi),\ a(p,\pi),\ \mu(p,\pi))$ be the unique optimal solution of the above f.o.c. relative to (p,π).[6] They are smooth functions of (p,π) under the assumptions on the utility function.

Remark 2

When financial opportunities are restricted to the subspace of R^N, we can restrict our attention to the following set of commodities and securities prices without losing anything in keeping track on the behavior of consumption demands

[6]See[2] for the behavior of the optimal solution with several budget constraints in detail.

due to the variant version of the homogenity property of the demand functions.[7]

$$P = \{p \in R^{(N+1)L}_{++} : \sum_c p^c(o) + \sum_c \sum_s p^c(s) = 1\}$$
$$Q(\pi) = \{\pi \in R^N_{++} : \sum_s \pi(s) = 1\}$$

Notice that $Q(\pi)$ is the set of normalized "no arbitrage" securities prices when financial opportunities are summarized by $\{A_h\}$.

Now, suppose that some $\pi \in Q(\pi)$ is fixed exogeneously. Then, it is convenient to define the following set for the later discussion:

$$O_N(A:\pi) = \{\sigma \in R^N_{++} : \sigma = (1/\sum_s \delta(s))\,(\delta(1),\ldots,\delta(N)) \text{ for } \delta \in O(A:\pi)\}$$

Lemma 1

$O_N(A:\pi)$ is an (N-M)-dim submanifold of $Q(\pi)$ for any $\pi \in Q(\pi)$ when A is an M-dim subspace of R^N.

Proof

First, $O(A:\pi)$ is an (N-M)-dim submanifold of R^N_{++} as the intersection of $T(A:\pi)$ and R^N_{++}. Then, since the mapping $\phi : O(A:\pi) \rightarrow O_N(A:\pi)$ is diffeomorphic, $O_N(A:\pi)$ is also an (N-M)-dim submanifold of R^N_{++}. Moreover, $O_N(A:\pi)$ is a subset of $Q(\pi)$ by construction. Q.E.D.

Remark 3

It is not difficult to see that $\bigcup_{\pi \in Q(\pi)} O_N(A:\pi) \equiv Q(\pi)$. We can call $\sigma \in O_N(A:\pi)$ as a "renormalized" LM vector. There is no loss of generality in introducing this renormalization because all adjustments can be made in terms of commodity prices p for keeping track on the behavior of consumption demands.

III. REAL INDETERMINACY OF FINANCIAL EQUILIBRIA

1. Admissible Case with Variable Contingent Securities Prices

There is no problem in the existence of a financial equilibrium no matter what type of $\{A_h\}$ is given simply because the aggregate excess demand functions for commodities are well behaved as in the Arrow-Debreu economy. This is also true even when π is fixed exogeneously. But we have no make some restrictions on the collection of A_h, $\{A_h\}$ for the discussion about the indeterminacy of financial equilibria.

[7] Many different normalization conventions can be utilized. So, different convention will be utilized if necessary without loss of generality.

Definition 2

Financial opportunities summerized by $\{A_h\}$ are called "admissible" if there is always some consumer, say, h = 1 such that $\sum_{h\neq 1} A_h \subsetneqq A_1$ where $\sum_{h\neq 1} A_h$ is a direct sum of subspaces.

First, we will restrict our attention to this admissible cass.

Definition 3

A 4-tuple(p,π,x,a) is a Financial Equilibrium(FE) rlative to $\{A_h\}$ if

i) (x_h, a_h) is an optimal solution to consumer's maxinization problem relative to (p,π) and

ii) $\sum_h x_h^c(s) = \sum_h eh^c(s)$ for $s = 0,1,..,N \quad c = 1,.., L$

$\sum_h a_h(s) = 0$ for $s = 1,..,N$

For the further discussion, let's first establish the equivalence between the following two representations of consumers' maximization problems:

(I) Maximize $U_h(x_h)$

s.t. $p(o)\ (x_h(o)-e_h(o)) \leqslant -\sum_s \pi(s)\ a_h(s)$

$p(s)\ (x_h(s)-e_h(s)) \leqslant a_h(s)$ for $s = 1,\ldots, N$

and $x_h \in x_h,\ a_h \in A_h$ for $h = 1,\ldots,m$

(II) Maximize $U_1(x_1)$

s.t. $p(o)\ (x_1(o)-e_1(o)) + \sum_s \pi(s)\ p(s)\ (x_1(s)-e_1(s)) \leqslant 0$

and $x_1 \in X_q$

Maximize $U_h(X_h)$

s.t. $p(o)\ (x_h(o) - e_h(o)) \leqslant -\sum_s \pi(s)\ a_h(s)$

$p(s)\ (x_h(s)-e_h(s)) \leqslant a_h(s)$ for $s = 1,\ldots,N$

and $x_h \in X_h,\ a_h \in A_h$ for $h = 2,\ldots,m$

This is nothing but the application of the well known isomorphism between two representations of consumers' maximization used by Cass[5] and also by Duffie and Shafer[10]

Lemma 2

If (p,π,x,a) is a FE in (II), then it is also a FE in (I) with

$$a_1 = -\sum_{h\geqslant 2} a_h$$

Proof

This is obvious simply because (x_1,a_1) defined here is also feasible and optimal relative to (p,π) in (I). Q.E.D

Lemma 3

If (p,π,x,a) is a FE in (I), then there is (p',π', a') supporting x as a FE allocation in(II).

Proof

First, set (p',π',a') as follows:

$p'=(p(o), (\sum_s \delta_1(s))\ p\ (1),\ldots, (\sum_s \delta_1(N))p(N))$, $\pi'=\sigma \in O_N\ (A_1{:}\pi)$ and $a_h'=\sum_s \delta_1(s))\ a_h$ for $h\geqslant 2$ where $\pi'=\sigma_1$ is derived uniquely by the normalization of the optimal LM vector δ_1 supporting consumer 1's optimal solution(x_1, a_1) relative to (p,π).

Then, it can be seen easily that x_1 and (x_h, a_h'), for $h\geqslant 2$, are feasible and also optimal relative to (p', π') in (II) and hence (p',π',x,a') is a FE in (II). Q.E.D

So, there is no loss of generality in discussing the indeterminacy of equilibria in terms of (II). Now, the question is how the indeterminacy of financial equilibria depends upon financial opportunities represented by $\{A_h\}$. Since financial markets clear always whenever commodities clear, we can restrict our attention to the behavior of the aggregate excess demand functions for commodities.

Let f_h: $P\times Q\ (\pi)\ \times\ R^{(N+1)L}_{++} \to R^{(N+1)L}_{+}$ denote the smooth demand function such that $f_h\ (p,\pi,e_h)=x_h$ for $h=2,\ldots,m$. Consumer 1's demand function is denoted by g_1: $P\times Q\ (\pi)\times R^{(N+1)L}_{++} \to R^{(N+1)L}_{+}$ such that $g_1\ (p,\pi,e_1)=x_1$.

Then, let's define the aggregate excess demand function $Z{:}P\times Q(\pi\times Q \to R^{(N+1)L-1}$ such that: $Z(p,\pi,e)=g_1\ (p,\pi,e_1)\ +\ \sum_{h\geqslant 2} f_h\ (p,\pi,e_h)\ -\ \sum_{h\geqslant 1} e_h$ with the 1_{st} commodity in $t=0$ dropped out by the analogue of Walras' law. Let's define the following equilibrium manifold:

$E=\{(p,\pi,e) \in \Pi{\neq}\Omega\ [\pi)\times\Omega{:}\ Z\ (p,\pi,e)=0\}$.

Lemma 4

E is an $[(N\text{-}1)+(N+1)Lm]$-dim submanifold of $PxQ(\pi)x\Omega$.

Proof

As in the standard regular economy, it is easy to see that rank $[\partial Z(p,\pi,e)/\partial e]=\text{rank}\ [\partial(g_1(p,\pi,e_1)\text{-}e_1)/\partial e_1]=(N+1)L\text{-}1$. Therefore, the Jacobian of the derivative mapping $DZ(p,\pi,e)$ evaluated at any (p,π,e) has a maximal rank and hence 0 is a regular value of the mapping Z. So, $E=Z^{-1}(0)$ is an $[(N\text{-}1)+(N+1)Lm]$-dim manifold by the preimage theorem. Q.E.D.

Now, define a mapping $\Gamma:E \rightarrow Q(\pi)x\Omega$ as a restriction of the natural projection to E. Since Γ is a smooth mapping between manifolds of the same dimension, which is also surjective and proper, the set of regular values of Γ is an open and dense subset of $Q(\pi)x\Omega$ by Sard's Theorem. The surjectivity of Γ is obvious because the equilibrium price correspondence E_P: $Q(\pi)x\Omega \rightarrow P$ such that

$E_p(\pi,e) = \{p \in P: Z(p,\pi,e) = 0\}$ is closed since Z is continuous.

The properness of Γ is also obvious with smooth demand functions and the behavior of consumer 1 on the boundary of P in particular (see [3] for the exhaustive analysis of these properties or [8] of which argument can be extended to the case here straightforwardly).

Definition 4

V is an open and dense subset of $Q(\pi)x\Omega$, composed of regular values of the mapping Γ.

The indeterminacy in this economy with an arbitrary admissible $\{A_h\}$ depends upon whether we can establish a well defined mapping("diffeomorphism" for instance) between a subset of V(and hence a subset of $PxQ(\pi)$) and a subset of equilibrium allocations when "e" is given. First, let's consider the following two "polar" cases:

Case 1: Uniform restriction on A_h

$A_h = A$ for $h \in H$ where "A" is an M-dim subspace of R^N spanned by M different mutual funds $k^1,..,k^M$. That is, all consumers are uniformly restricited to hold their portfolios only as some combinations of M mutual funds formed from contingent securities. Then, it is not surprising to see that this economy with $\{A_h\} = \{A\}$ is exactly isomorphic to the economy with incomplete financial markets where "M" non-contingent securities are traded without any restrictions.

Thus, the already well known real indeterminacy with incomplete financial markets can be exactly reproduced in this framework.

For instance, if π is supposed to be determined in financial markets, then it equivalent to the case examined by Geanakoplos and Mas-Colell [11] where the set of equilibrium allocations was shown to contain an (N-1)-dim manifold when returns from financial assets are fixed, but assets prices are variables.

Case 2: No restriction on A_h

$A_h = R^N$ for $h \in H$. This is the case with a complete set of contingent securities and without restrictions. Thus, there is only nominal indeterminacy in the space of normalized prices, corresponding to every $\pi \in Q(\pi)$.

Next, it may not be easy for us to draw a systematic conclusion about the indeterminacy of financial equilibria relative to any arbitrary admissible $\{A_h\}$. But, there may be some class of $\{A_h\}$, besides those special cases mentioned previously, that yields real indeterminacy of equilibria with the dimensional property.

For this, let's define the set of equilibrium allocations when some $\{A_h\}$ and "e" are given:

$X(e)=\{x\in Q$: there is $(p,\pi,e) \in \Gamma^{-1}(\pi, e)$ such that
$x=(g_1(p,\pi,e_1), f_2(p,\pi,e_2),\ldots,f_m(p,\pi,e_m))$ for $(\pi,e) \in \{Q(\pi)x\{e\}\}\cap V\}$

Here, it is convenient for us to examine the case without restrictions, i.e., $A_h=R^N$ for all h.

Definition 5

A FE(p,π,x,a) with $A_h=R^N$ for all h is called a Walrasian Equilibrium (WE), which is always Pareto optimal.

Since there is only nominal indeterminacy in the space of prices and in portfolio choices, let (p^*, π^*, x^*, a^*) denote a WE corresponding to $\pi^*=(1/N,..,1/N) \in Q(\pi)$ in particular. Then, there exists (p,π,a), supporting x^* as a WE, with the following properties:

$p=(p^*(0), (1/N\pi(1))p^*(1),..,(1/N\pi(N))p^*(N))$, $\pi \in Q(\pi)$ and
$a_h=((1/N\pi(1))\, a_h^*(1),.., (1/N\pi(N)a_h^*(N))$ for $h \in H$.

Now, let's define $A_h(\pi)$ as follows:

$A_h(\pi)=\{a_h \in R^N$: $a_h=((1/N\pi(1))\, a_h^*(1),\ldots, (1/N\pi(N))\, a_h^*(N))$ for $\pi \in Q(\pi)\}$

Obviously, $A_h(\pi)$ is an (N-1)-dim submanifold of R^N since the mapping from $Q(\pi)$ to $A_h(\pi)$ is diffeomorphic. Notice that $A_h(\pi)$ will be defined for each Pareto optimal allocation if there are a finite number of WE allocations. Notice also that $A_h(\pi)$ is independent of the choice of π^* from $Q(\pi)$.

Lemma 5

Suppose that(p,π,x,a) is a FE relative to any admissible $\{A_h\}$. Then, it is Pareto optimal if and only if $a_h \in A_h(\pi)$ for $h \in H$.

Proof

This is obvious from the construction of $A_h(\pi)$. Q.E.D.

Next, financial opportunities are assumed to satisfy the following additional condition.

(a1) there is a non-empty subset $H_s \subset H$ such that $\dim(A_h)<N$ for $h \in H_s$.

Now, define the following set:

$V_1=\{(\pi,e) \in V$: there is $(p,\pi,e) \in \Gamma^{-1}(\pi,e)$ such that $x(p,\pi,e)$ is not Pareto optimal$\}$

Lemma 6

If $\{Ah\}$ satisfies (a1), then V_1 is an open and dense subset of V and hence of $Q(\pi)x\Omega$.

Proof

1) openness: Suppose that $x\ (p,\pi,e)$ is not Pareto optimal relative to $(\pi,e) \in V$. This implies that there is some $h \in H_s$ such that $a_h \in A_h \cap A_h(\pi)$ and also $\delta_h = \pi$. Since a_h and δ_h are continuous with respect to (π,e), there is an open neighborhood $V(\pi,e)$ of (π,e) in V such that $x(p',\pi',e')$ is not Pareto optimal relative to $(p', \pi',e') \in \Gamma^{-1}\ (\pi', e')$ for $(\pi', e') \in V\ (\pi,e)$.

2) denseness: Since $A_h(\pi)$ is constructed after the collinearity in a_h is eliminated by the normalization of $\pi \in Q(\pi)$, $A_h \cap A_h(\pi)$ must be at most a closed and nowhere dense subset of A_h for $h \in H_s$. Now, suppose that $x(p,\pi,e)$ is a Pareto optimal allocation relative to $(p,\pi,e) \in \Gamma^{-1}\ (\pi,e)$ for $(\pi,e) \in V$. Then, we can pick $(\pi', e) \in V$ such that $\|(\pi', e) - (\pi,e)\| < \varepsilon$ for some arbitrary small $\varepsilon > 0$ and $a_h' \notin A_h \cap A_h(\pi)$ for some $h \in H_s$ where $\{a_h'\}$ is the collection of equilibrium portfolios supporting $x(p',\pi',e)$ relative to $(p',\pi',e) \in \Gamma^{-1}\ (\pi', e)$.

Then, $x\ (p',\pi',e)$ is not Pareto optimal by Lemma 5. Q.E.D.

Next, let's define the following set:

$\Omega_1 = \{e \in \Omega$: there is $\pi \in Q(\pi)$ such that $(\pi, e) \in V_1\}$

Then, it is obvious that Ω_1 is an open and dense subset of Ω as a cross section of V_1 from Lemma 6. Now, pick an open subset V_0 around $(\pi,e) \in V_1$. Since $\Gamma : E \to Q\ (\pi) x \Omega$ is a smooth mapping between manifolds of the same dimension and is also proper, $\Gamma^{-1}\ (\pi,e)$ is a finite set of dimension 0 for every $(\pi, e) \in V_0$. Then, there are a finite number of disjoint open (relative to E) subsets $W_1,\ldots,W_k$ of E, containing each point $(p,\pi,e) \in \Gamma^{-1}\ (\pi,e)$ such that Γ maps each W_j, $j = 1,..,k$, diffeomorphically on V_0 by the stack of record theorem. That is, the mapping Γ restricted to W_j and V_0 is a diffeomorphism.

Next, define the following sets:

$G = \{Q(\pi) x \{e\}\} \cap V_0$

$V(p,\pi) = \{(p,\pi) \in PxQ(\pi)$: there is $(p,\pi,e) \in \Gamma^{-1}\ (G)\}$

which is also an (N-1)-dim submanifold of $PxQ(\pi)$ by the restriction of the local diffeomorphism Γ^{-1} to G.

Then, $\Gamma^{-1}(G) = V(p,\pi) x \{e\} \quad W_j$.

Since $x(p,\pi,e)$ is Pareto optimal relative to $(\pi,e) \in V$ if and only if $\delta_h\ (p,\pi,e_h) = \pi$ for all $h \in H$, the following variant version of V_1 and Ω_1 is useful for the discussion.

$V_2 = \{(\pi,e) \in V$: there is $(p,\pi,e) \in \Gamma^{-1}\ (\pi,e)$ such that

$\delta_j\ (p,\pi,e_j) - \pi = v_j \in O\ (A_j) \setminus \{0\}$ for some $j \in H\}$

$Q_2 = \{e \in \Omega$: there is $\pi \in Q(\pi)$ such that $(\pi,e) \in V_2\}$

Notice that V_2 is a subset of V_1.

Lemma 7

V_2 is an open and dense subset of V and hence of $Q(\pi) x \Omega$.

Proof

The openness of V_2 follows from the fact that δ_j (and hence v_j) is continuous with respect to (p,π,e). For the denseness, suppose that $(\pi,e) \in V$, but $\notin V_2$. Let $x=(x_1,\ldots,x_m)$ be an equilibrium allocation relative to $(p,\pi,e) \in \Gamma^{-1}(\pi,e)$. Now, define $x'=(x_1,.., x_{j-1}, x_j', x_{j+1},.., x_m)$ such that $\|x_j'-x_j\| < \varepsilon$ for any arbitrary small $\varepsilon > 0$ and x_j' together with $a_j' \in A_j$ and μ_j' is an opimal solution to the equations of the first order conditins (2) and (3) when $M=N-1$ relative to (p,π,e_j') where $e_j'=e_j+(x_j'-x_j)$ and $\delta_j'=(\mu_j'(1)/\mu_j'(0),\ldots,\mu_j'(N)/\mu_j'(o))=\pi+v_j'$ for $v_j' \in O(A_j)\setminus\{0\}$. Notice that such (x_j', a_j', μ_j') exists always due to the assumption (u3) on the utibility function and hence due to the non-singularity of the Jacobian of the system of equations (2) and (3). Now, pick $(\pi', e') \in V$ such that $\pi'=\pi$ and $e'=e+(x'-x)$, which implies that $\|(\pi', e')-(\pi,e)\|=\|x_j'-x_j\| < \varepsilon$ for any $\varepsilon > 0$. Then, x' is an equilibrium allocation relative to $(p,\pi,e') \in \Gamma^{-1}(\pi,e')$ and hence $(\pi,e') \in V_2$. So, V_2 is open and dense in V.

Proposition 1

If $A_1=R^N$ and there is some $j \in H$ such that $\dim(A_j)=N-1$ with the origin as the only element of $O(A_j)$ containing 0 as its corrdinate, then $X(e)$ contains an $(N-1)$-parameter family of equilibrium allocations for every $e \in \Omega_2$ no matter which A_h is given to $h \in H\setminus\{1,J\}$.

Proof

Pick any arbitrary (p,π), $(p',\pi') \in V(p,\pi)$ supporting x and x′ respectively. Then, it is sufficient to show that $x \neq x'$.

Suppose not, i.e., $x=x'$. Then, we'll get the following relations from the f.o.c. evaluated at $x_h=x_h'$ after using the normalization

$p^1(o)=p^1(o)'=1$:

(5) $p(o)=p(o)'$, $\mu_1(o)'$, $\pi(s)p(s)=\pi(s)'p(s)'$ for $h=1$

(6) $\mu_j(o)=\mu_j(o)$, $\delta_j(s)p(s)=\delta_j(s)'p(s)'$ for $h=j$

(7) Hence, $\delta_j(s)'=(\pi(s)'/\pi(s))\delta_j(s)$ for $s=1,..,N$ from (5),(6)

$$\begin{aligned}\text{Then, } (\delta_j'-\pi') &= (v_j(1)',\ldots,v_j(N)') \\ &= (\delta_j(1)'-\pi(1)',\ldots, \delta_j(N)'-\pi(N)') \\ &= ((\pi(1)'/\pi(1))\delta_j(1)-\pi(1)',\ldots, (\pi(N)'/\pi(N))\delta_j(N) - \pi(N)') \\ &= ((\pi(1)'/\pi(1))v_j(1),\ldots, (\pi(N)'/\pi(N))v_j(N)) \in O(A_j)\end{aligned}$$

(8) from (7) and $(\delta_j\text{-}-)=(v_j(1),\ldots, v_j(N)) \in O(A_j)$

Obviously, $v_j \neq 0$ and $v_j' \neq 0$ by the construction of $V(p,\pi)$.

Now, from(8) and the property of $O(A_j)$, it must be the case that there is some scalar $k \neq 0$ such that $v_j'=k\cdot v_j$. Then, it follows immediately that

(9) $\pi'(1)+\ldots+\pi'(N)=k(\pi(1)+\ldots+\pi(N))=k=1$

This implies that $v_j'=v_j$ and hence $\pi'=\pi$. Then, $p=p'$ from (5).

Hence$(p,\pi)=(p',\pi')$, which is a contradiction. Thus, there is a 1-to-1 correspondence between $V(p,\pi)$ and the suitably defined subset of $X(e)$.

Let $\tilde{X}(e)$ denote such a subset of $X(e)$. Then, the mapping $F_v{:}V(p,\pi)\rightarrow\tilde{X}(e)$ is continuously differentiable by the implicit function theorem, 1-to-1 and onto by construction. Therefore, $X(e)$ contains an (N-1)-parameter family of equilibrium allocations for every $e \in \Omega_2$. Q.E.D.

Next, a broader class of admissible $\{A_h\}$ can be considered.

Although it can be conjectured that almost any admissible $\{A_h\}$ would be associated with a non-trivial real indeterminacy of equilibria, we must be precise in characterizing the conditions under which there is a dimensional real indeterminacy. For this, the assumptions on $\{A_h\}$ are modified a little bit in the following way:

(a2) $\{A_h\}$ is admissible with $\dim(A_1)=M$ shch that $1\leqslant M<N$.

(a3) $\{A_h\}$ is diverse enough so that there is a subset of consumers $H_M \subset H\backslash\{1\}$ with $A_1 = \sum_{h\in H_M} A_h$ and $A_i\neq A_j$ for some $i,j \in H_M$ with $\#(H_M)=M$.

(a4) $m>M$

We can set $H_M=\{2,..,M+1\}$ without loss of generality.

Now, it is easy to see that all previous arguments about the equilibrium manifold and the projection mapping hold for the case here. So, E and the mapping Γ are defined as before together with an open and dense subset V of $Q(\pi)x\Omega$ as the set of regular values of Γ.

Next, define the following set V_3:

$V_3=\{(\pi,e) \in V$: there is $(p,\pi,e) \in \Gamma^{-1}(\pi,e)$ such that rank $([a]_M)=M\}$ where $[a]_M$ is an NxM matrix of equilibrium portfolios relative to $(p,\pi,e) \in \Gamma^{-1}(\pi,e)$ such that $a_h=(a_h(1),\ldots,a_h(N))$ is the h_{th} column vector for $h \in H_M$ and $a(s)=(a_2(s),\ldots,a_{M+1}(s))$ is the s_{th} row with $a(s)\neq 0$ for $s=1,..,N$.

Lemma 8

V_3 is an open and dense subset of V and hence of $Q(\pi)x\Omega$.

Proof

The straightforward application of the steps in [5] is sufficient for the proof here. This application is possible due to the assumption

(a2) - (a4). See p.18-20 of [5] for the detail. Q.E.D.

So, $V(p.\pi)$ is defined again as an (N-1)-dim submanifold of $PxQ(\pi)$ for every $e \in \Omega_3$ defined in line with Ω_1, Ω_2 by the stack of record theorem as before with rank $([a]_M)=M$ for every$(p,\pi) \in V(p,\pi)$. $X(e)$ is also defined as the set of equilibrium allocations relative to $e \in \Omega_3$.

Lemma 9

Let $Y([a]_M)$ denote an M-dim subspace of R^N, spanned by column vectors of $[a]_M$ (i.e., $Y([a]_M) = \sum_{h \in H_M} A_h$) and O $([a]_M)$ denote its orthogonal complement. Then, $Y([k/g]\ [a]_M) \cap O([a]_M) = \{0\}$ where $[k/g]$ is a diagonal matrix with $(k(1)/g(1),\ldots,k(N)/g(N)) >> 0$ on the main diagonal.

Proof

Suppose not. Then, there is some $b' \in R^M$ such that $[k/g]\ [a]_M.\ b' \in O([a]_M)$ with $b' \neq 0$. Then, $([k/g]\ [a]_M.b')^T\ y = 0$ for every $y \in Y\ ([a]_M)$ by the orthogonality where $y = [a]_M.\ b$ for $b \in R^M$. So, pick some $y' = [a]_M b'$. Then, $([k/g]\ [a]_M.b')^T y' = ([k/g]\ [a]_M.b')^T\ [a]_M b' = ([a]_M b')^T\ [k/g]([a]_M b') > 0$, which is is a contradiction. Q.E.D.

Lemma 10

$Y([a]_M) \equiv Y([k/g]\ [a]_M)$ if and only if $k = g$.

Proof

First, every column vector in $[k/g]\ [a]_M$ can be decomposed as follows: $a_2 + [k/g) - 1]a_2,\ldots\ldots,a_{M+1} + ([k/g]-1]a_{M+1}$

where $[(k/g)-1] = [k/g]-I_N$ and $1 \in R^N$

Since the sufficiency part is obvious, let's consider the necessity part. Suppose that $Y([a]_M) \equiv Y([k/g]\ [a]_M)$, but $k \neq g$.

Then, $[k/g]a_h - a_h = [(k/g)-1]a_h \in Y([k/g] \cap O\ ([a]M)$ for $h \in H_M$ by the orthogonal decomposition. This implies that $[(k/g)-1]a_h = 0$ for all $h \in H_M$ by Lemma 9. Then, $k = g$ by the property of $[a]_M$, which is a contradiction. Q.E.D.

Proposition 2

If $\{A_h\}$ satisfies (a2)-(a4), then X(e) contains an (N-1)-parameter family of equilibrium allocations for every $e \in \Omega_3$.

Proof

Let x and x′ denote equilibrium allocations relative to (p,π), $(p',\pi') \in V(p,\pi)$. Suppose that $x = x'$. Then, from the f.o.c. of consumer's maximization, the following relation must hold:

$\delta_h(s)p(s) = \delta_h(s)\ (p^1(s),\ldots,\ p^L(s)) = \delta_h(s)'\ (p^1(s)',\ \ldots\ ,\ p^L(s)') = \delta h(s)'\ p(s)'$

(10) which implies that $\delta_h(s)'/\delta_h(s) = p^c(s)/p^c(s)'$

for $h \in H_M$, $s = 1,\ldots,N$ and $c = 1,\ldots,L$

(11) Also, from (10) and $x_h = x_h'$, $[a]_M = [p^c/p^{c\prime}]\ [a']_M$

where $[a]_M$, $[a']_M$ are matrices of equilibrium portfolios supporting x_h, x_h' for h

$\in H_M$ and $[p^c/p^{c\prime}]$ is a diagonal matrix with $(p^c/p^{c\prime}) = (p^c(1)/p^c(1)', \ldots, p^c(N)/p^c(N)')$ on the main diagonal.

(12) Thus, $Y([a]_M) \equiv Y([p^c/p^{c\prime}]\,[a']_M) \equiv Y([a']_M) \equiv A_1 \equiv \sum_{h \in H_M} A_h$

(13) Lemma 10 and (12) implies that $p = p'$ and $\delta_h = \delta_h'$ for $h \in H_M$

Next, $\pi \neq \pi'$ from (13) since $(p,\pi) \neq (p',\pi')$ by assumption. This implies that $T(A_h{:}\pi) \cap T(A_h{:}\pi') = \emptyset$ for at least some $h \in H_M$ because they are parallel to each other. This is a contradiction to $\delta_h = \delta_h'$ for at least some $h \in H_M$. Therefore, $\pi = \pi'$ and hence $(p,\pi) = (p',\pi')$, which is also a contradiction. Thus, the mapping such that $F_v{:}V(p,\pi) \to \tilde{X}(e)$ with $F_v(V(p,\pi)) = \tilde{X}(e)$ is is continuously differentiable, 1-to-1 and onto by construction. Therefore, X(e) contains an (N-1)-parameter family of equilibrium allocations for every $e \in \Omega_3$. Q.E.D.

The following example will be useful to understand the above proposition.

[Example 1]

There are 3 consumers, $h = 1,2,3$. there is only one commodity in each period and there are 3 uncertain states in period 1, $s = \alpha,\beta,\tau$

Each consumer's endowment, denoted by $e_h = (e_h(o), e_h(\alpha), e_h(\beta), e_h(\tau))$, is assumed to satisfy that:

there is no "t" $\in R$ such that $DU_i(e_i) = t.DU_j(e_j)$ for any i,j

Financial opportunities are assumed to be given as follows:

$A_1 = \{a_1 \in R^3{:}\ a_1 = b_1 \cdot v + b_2 \cdot u$ for some $b_1, b_2 \in R\}$ where two vectors $v = (v(\alpha), v(\beta), v(\tau))$ and $u = (u(\alpha), u(\beta), u(\tau))$ are linearly independent

$A_2 = \{a_2 \in R^3{:}\ a_2 = k_2 \cdot v$ for some $k_2 \in R\}$
$A_3 = \{a_3 \in R^3{:}\ a_3 = k_3 \cdot u$ for some $k_3 \in R\}$

which implies that $A_1 = A_2 + A_3$.

Also, $V(p,\pi)$ is defined as a 2-dimensional submanifold here.

Now, let(p,π,x,a) and (p',π',x',a') be FE relative to (p,π), $(p',\pi') \in V(p,\pi)$. Also, suppose that $x = x'$. From the f.o.c. of consumer's maximization, the following relationship must hold (we can set $p(o) = p(o)' = 1$ without loss of generality):

(14) $\mu h(o) = \mu(o)'$, $\delta_h(s)\, p\,(s) = \delta_h(s)' p(s)'$ and $\delta_h(s)\, a_h(s) = \delta_h(s)' a_h(s)'$
for $h = 1,2,3$ and $s = \alpha,\beta,\tau$

(15) From (14), $a_h(s) = (\delta_h(s)'/\delta_h(s))\, a_h(s)' = (p(s)/p(s)')\, a_h(s)'$

We can express (15) by using the matrix notation as follows:

$$\begin{bmatrix} a_2(\alpha) & a_3(\alpha) \\ a_2(\beta) & a_2(\beta) \\ a_2(\tau) & a_3(\tau) \end{bmatrix} = \begin{bmatrix} p(a)/p(a)' & 0 & 0 \\ 0 & p(\beta)/p(\beta)' & 0 \\ 0 & 0 & p(\tau)/p(\tau)' \end{bmatrix} \begin{bmatrix} a_2(\alpha)' & a_3(\alpha)' \\ a_2(\beta)' & a_3(\beta)' \\ a_2(\tau)' & a_3(\tau)' \end{bmatrix}$$

or simply $[a]_2 = [p/p'] \ [a']_2$

Notice that rank $([a]_2) = \text{rank}([a']_2) = 2$ by the property of A_2 and A_3.

Thus, $Y([a]_2) \equiv Y([p/p'] \ [a']_2) \equiv Y([a']_2) \equiv Y(A_1) \equiv Y(A_2 + A_3)$.

(16) So, $p = p'$ by Lemma 10 and hence $\delta_h = \delta_h'$ from(14)

Since $\delta_h \in T(A_h:\pi)$ and $\delta_h' \in T(A_h:\pi')$ and $T(A_h:\pi) \cap T(A_h:\pi') = \varnothing$ for at least some h(here $h = 1$ in particular) if $\pi \neq \pi'$, this is a contradiction to (16). Hence, $(p,\pi) = (p',\pi')$, which is also a contradiction.

Thus, $x \neq x'$. Therefore, there is a 2-parameter family of equilibrium allocations corresponding to each$(p,\pi) \in V(p,\pi)$.

Remark 3

Proposition 2 is indeed the generalization of the main results in Cass and Balasko[3] and Geanakoplos and Mas-Colell [11] because Lemma 8, 9, 10 and Proposition 2 hold when $\{A_h\}$ is reduced to a simple one such that $A_h = A$ for $h \in H$ where A is an M-dim subspace of R^N.

2. Admissible Case with Fixed Contingent Securities Prices

We will consider the case with fixed $\pi^* \in Q(\pi)$ in this section. Suppose that (p,π^*,x,a) is a FE in (I) relative to some $\pi^* \in Q(\pi)$. $O(A_h:\pi^*)$ and $O_n(A_h:\pi^*)$ are defined as before.

Lemma 11

$\bigcap_{h \in H} O_n(A_h:\pi^*)$ is non-empty for any $\pi^* \in Q(\pi)$ relative to any admissible.

Proof

This is also obvious simply because π^* is always in $O_N(A_h:\pi^*)$ for all $h \in H$. Therefore, $\bigcap_{h \in H} O_N(A_h:\pi^*) = \varnothing$ for $\pi^* \in Q(\pi)$. Q.E.D.

Let $T(\pi^*) = \bigcap_{h.H} O_N(A_h:\pi^*)$. Then, $T(\pi^*) = O_N \ (A_1:\pi^*)$ by the property of an admissible $\{A_h\}$. Notice that $T(\pi^*)$ is always a submanifold of $Q(\pi)$ regardless of the choice of $\{A_h\}$ and π^*.

Lemma 12

If (p',π',x',a') is a FE in (II) relative to some $\pi = \sigma \in T(\pi)$, then there exists a FE (p,π^*,x,a) in (I) such that:

$p=(p(o)', (1/\sum_s \delta(s))p(1)',\ldots, (1/\sum_s \delta(s))p(N)')$ and $x_h = x_h'$,

$a_h = (1/\sum_s \delta(s))a_h'$ for all h where $\sigma = (1/\sum_s \delta(s))\ (\delta(1),\ldots,\ \delta(N))$.

Proof

By Lemma 2, (p',π',x',a') is also a FE in (I) with $a_1' = -\sum_{h\geq 2} a_h'$.

Now, take a look at the budget constraints of all consumers with the following manipulation:

$$\begin{aligned} p(o)(x_h(o)-e_h(o)) = \ & p(o)'(x_h(o)'-e_h(o)) = -\sum_s \pi(s)'a_h(s)' = -\sum_s \sigma(s)a_h(s)' \\ & = -\sum_s [\delta(s)/\sum_s(s))\ a_h(s)' = -\sum_s (v(s)+\pi(s)^*)\ (a_h(s)'/\sum_s \delta(s)) \\ & = -\sum_s \pi(s)^*\ (a_h(s)'/\sum_s \delta(s)) = -\sum_s \pi(s)^*\ a_h(s) \end{aligned}$$

$$p(s)\ (X_h(s)-e_h(s)) = (1/\sum_s \delta(s))p(s)'\ (x_h(s)'-e_h(s)) = (1/\sum_s \delta(s))a_h\ (s)' = 2_h(s)$$

for $s=1,..,N$ and $h=1,..,m$. It is also easy tio verify that all the other conditions for coinsumer's maximization are satiafied.

Thus, (x_h,a_h) is feasible and also optimal relative to (p,π^*) Q.E.D.

Proposition 3

The set of equilibrium allocations with a fixed $\pi^* \in Q(\pi)$ in (I) is identical to the set of equilibrium allocations with π varying within $T(\pi^*)$ in (II).

Proof

This is obvious from Lemma 11 and Lemma 12. Q.E.D.

Now, let's define the following set again:

$E(\pi^*) = \{(p,\pi,e) \in P x T(\pi^*) x \Omega\text{: } Z(p,\pi,e) = 0\}$

which is nothing but an equilbrium manifold E when $Q(\pi)$ is restricted to $T(\pi^*)$. Notice that the dimension of $E(\pi^*)$ is dependent on the choice of $\{A_h\}$ unlike E and hence the indeterminacy of equilibria is too.

Proposition 4

if π is fixed at some $\pi^* \in Q(\pi)$ and $\dim(T(\pi^*)) = n$, then X(e) contains an n-parameter family of equilibrium allocations for every $e \in \Omega_3$.

Proof

This is obvious because all the previous arguments about real indeterminacy are still valid after E is replaced by $E(\pi^*)$. Q.E.D.

Remark 4

When $\pi^* \in Q(\pi)$ is fixed and $\{A_h\}$ is given with $A_1 = R^N$, then there exists at most a finite number of financial equilibrium allocations no matter which A_h is given for $h \in H\backslash\{1\}$ since $T(\pi^*)=\pi^*$. Also, when A_h is given with $\dim(A_1)=M$, then $\dim (T(\pi^*))=N\text{-}M$ and hence ther exists an (N-M)-parameter family of equilibrium allocations if $\{A_h\}$ satisfies (a2)-(a4).

IV. ROBUSTNESS OF FINANCIAL EQUILIBRIA

Financial economy considered here exhibits the striking property of dimensional real indeterminacy which does not appear in the Arrow-Debreu economy. In addition to this property, as was pointed out by Geanakoplos and Mas-Colell[11], the complete market hypothesis with financial assets lacks "robustness" because there appears an (N-1)-dimensional real indeterminacy even after only a single financial asset is missing relative to the number of states to be insured.

This lack of robustness can be more conspicuously demonstrated here.

Proposition 5

If $\{A_h\}$ is given such that $A_h=R^N$ for all $h \in H\backslash\{j\}$ and A_j with $\dim(A_j)=N\text{-}1$ and with the origin as the only element of $O(A_j)$ containing 0 as its coordinate, then $X(e)$ contains an (N-1)-parameter family of equilibrium allocations for every $e \in \Omega_1$.

Proof

Under the assumption on $\{A_h\}$ here, $V_1=V_2$. Hence, all the steps in the proof of proposition 1 will hold for every $e \in \Omega_1$. Therefore, there appears an (N-1)-parameter family of equilibrium allocations. Q.E.D.

This implies that real indeterminacy will arise even with complete financial markets if there is just a single consumer who is minimally restricted to choose his portfolio of contingent securities in financial markets. Also, the set of financial equilibrium allocations may be quite volatile to a little perturbation of financial opportunities and this can be interpreted as another aspect of the lack of robustness of financial economy. The following examples will be useful to understand these arguments.

[Example 2]

There are 3 consumers $h=1,2,3$ and two uncertain states in period 1, denoted by $s=\alpha,\beta$. There is only one commodity in each period.

There are two contingent securities available in the current financial markets. Consumers' endowments are assumed to be given as follows: $e_h=(e_h(o), e_h(\alpha), e_h(\beta)) \in R^3_{++}$ and there is no "t" $\in R_+$ such that $DU_i(e_i)=t\cdot DU_j(e_j)$ for any i,j

Also, financial opportunities are assumed to be given as follows:

$A_1=A_2=R^2$

$A_3=\{a_3 \in R^2\text{: } a_3=(a_3\ (\alpha), a_3(\beta))$ such that $a_3(a)=k\cdot a_3(\beta)$ for some $k=0\}$

which is a 1-dimensional subspace of R^2, parameterized by the choice of a real number k.[8)]

Here, the following normalization convention is utilized for the notational convenience:

$P = \{p \in R^3_{++}: p(o) = 1\} = \{1\}xR^2_{++}$ and
$Q(\pi) = \{\pi \in R^2_{++}: \pi(\alpha) = 1\} = \{1\}xQ(\pi(\beta)) = \{1\}xR_+$

Next, define the following set:
$S(\pi(\beta)) = \{\pi(\beta) \in R$: there is $p \in P$ and $a_h \in A_h$ shch that (p,π,x,a) is a Pareto optimal FE$\}$

Since there will be generically a finite, odd number of Pareto optimal FE relative to the given endowment, there will be at most a finite, odd number of $\pi(\beta) \in Q(\pi(\beta))$ supporting Pareto optimal FE.

Thus, $S(\pi(\beta))$ contains at most a finite, odd number of elements.[9)]

Notice that $Q(\pi(\beta))\backslash S(\pi(\beta))$ is the analogue of $V(p,\pi)$ in this example.

Now, let's take a look at the first order conditions of consumers' maximization problem:

(17) $DU_h(x_h) = (\mu_h(o),\ \mu_h(\alpha)\ p(\alpha),\ \mu_h(\beta)\ p(\beta))$
$\mu_h(o) = \mu_h(\alpha),\ \mu_h(o)\ \pi(\beta) = \mu_h(\beta)$ for $h = 1,2$
(18) $DU_3(x_3) = (\mu_3(o),\ \mu_3(\alpha)\ p(\alpha),\ \mu_3(\alpha),\ \mu_3(\beta)\ p(\beta)$
$\mu_3(o)\ (k + \pi(\beta)) = k\mu_3(\alpha)\ + \mu_3(\beta)$ for $h = 3$

Next, pick $\pi(\beta), \pi(\beta)' \in Q\ (\pi(\beta))\backslash S(\pi(\beta)$ such that $\pi(\beta) \neq \pi(\beta)'$ and let (p,π,x,a) and (p′,π′,x′,a′) be corresponding FE.

Suppose that $x = x'$. Then the following relations must hold:

(19) $\mu_h(o) = \mu_h(o)',\ p(\alpha) = p(\alpha)',\ \pi(\beta)\ p(\beta) = \pi(\beta)'\ p(\beta)'$ from (17) for $h = 1,2$ and
(20) $\mu_3(o) = \mu_3(o)',\ \delta_3(\alpha) = \delta_3(\alpha)',\ \delta_3(\beta)\ p\ (\beta)'$ from (18) and
$k + \pi(\beta) = k\delta_3(\alpha) + \delta_3(\beta),\ k + \pi(\beta)' = k\delta_3(\alpha)' + \delta_3(\beta)$

Next, multiplying $p(\beta)$ to both sides of the 1_{st} relation and $p(\beta)'$ to the 2_{nd} relation in(20) and using(19), we can get the following relation:

(21) $kp(\beta) + \pi(\beta)\ p\ (\beta) = k\delta_3\ (\alpha)\ p\ (\beta)\ +\ \delta_3\ (\beta)\ p\ (\beta)$
(22) $kp(\beta)' + \pi(\beta)'\ p(\beta)' = kp\ (\beta)' + \pi(\beta)p(\beta) = k\delta_3(\alpha)'\ p(\beta)' + \delta_3(\beta)'\ p(\beta)'$
$= k\delta_3(\alpha)p(\beta)' + \delta_3(\beta)p(\beta)$
(23) Substracting (22) from (21), $k(p(\beta)\text{-}p(\beta)') = k\delta_3(\alpha)\ (p(\beta)\text{-}p(\beta)')$

[8]In this case, it amounts to saying that he is subject to hold only a mutual fund formed from contingent securities α,β in the ratio of $(k/(1 + k),\ 1/(1 + k))$.

[9]$S(\pi(\beta))$ may be empty according to the choice of k.

Since $k=0$ and $p(\beta)=p(\beta)'$, $\delta_3(\alpha)=1$ from (23) and $\delta_3(\alpha)'=1$ from (20). Also $\delta_3(\beta)=\pi(\beta)$ and $\delta_3(\beta)=\pi(\beta)$ from (20) and hence $\pi(\beta)$, $\pi(\beta)' \in S\ (\pi(\beta))$. This is a contradiction. Thus, $x \neq x'$. Therefore, there exists a 1-parameter family of distinctive financial equilibrium allocations corresponding to each $\pi(\beta) \in Q(\pi(\beta))$.

Observe that this argument is valid whether $S(\pi(\beta))$ is empty or not. If $S(\pi(\beta))$ is empty, then there is a 1-parameter family of distinctive equilibrium allocations and none of them is Pareto optimal. If $S(\pi(\beta))$ is non-empty, then there is a 1-parameter family of distinctive equilibrim allocations converging to a Pareto optimal allocation.

[Example 3]
Suppose that other fundamentals of the economy are identical to example 2 except financial opportunities. Here, two types of financial opportunities will be compared with each other. Let $\{A_h\}$ and $\{A_h'\}$ denote two different financial opportunities in the following sense:

$\{A_h\}$; $A_1 = R^2$
$A_2 = \{a_2 \in R^2\colon a_2(\alpha) = k \cdot a_2(\beta) \text{ for } k \neq 0\}$
$A_3 = \{a_3 \in R^2\colon a_3(\alpha) = g \cdot a_3(\beta) \text{ for } g \neq 0,\ k\}$
$\{A_h'\}$; $A_1' = A_1$
$A_2' = A_2$
$A_3' = \{a_3 \in R^2\colon a_3(\alpha) = g' \cdot a_3(\beta)$ for $g' = g + \varepsilon$ where ε is an arbitrary small real number$\}$

Let (p,π,x,a) be a FE relative to $\{A_h\}$ when $\pi(\beta) \in Q(\pi(\beta))\backslash S(\pi(\beta))$. Now, suppose that x is also a FE allocation relative to $\{A_h'\}$.
Then, there must be (p',π',a'), supporting x, satisfying the following conditions:

(24) $DU_h(x_h) = (\mu h(o),\ \mu h(\alpha)p(\alpha),\ \mu_h(\beta)p(\beta))$
$= (\mu_h(o)',\ \mu_h(\alpha)'p(\alpha)',\ \mu_h(\beta)'p(\beta)')$
(25) $\mu_{(0)}\ \mu_1(\alpha)$, $\mu_1(o)' = \mu_1(\alpha)'$ and hence $\delta_1(\alpha) = \delta_1(\alpha)'$
$\delta_1(\beta) = \pi(\beta)$, $\delta_1(\beta)' = \pi(\beta)'$ and hence $\pi(\beta)\ p\ (\beta) = \pi(\beta)'p(\beta)'$
(26) $k + \pi(\beta) = k\delta_2(\alpha) + \delta_2(\beta)$
$g + \pi(\beta) = g\delta_3(\alpha) + \delta_3(\beta)$ and
(27) $k + \pi\ (\beta)' = k\delta_2\ (\alpha)' + \delta_2(\beta)'$
$g' + \pi(\beta)' = g'\delta_3(\alpha)' + \delta_3(\alpha)' + \delta_3(\beta)'$
(28) From (24), (25), $p(\alpha) = p(\alpha)'$ and hence $a_h(\alpha)'$ for $h = 1,2,3$
Then, $\alpha_2(\alpha) = ka_2(\beta) = a_2(\alpha)' = ka_2(\beta)'$ and hence $a_2(\beta) = a_2(\beta)'$
(29) Thus, $p(\beta) = p(\beta)'$ from (28) which implies that $a_3(\beta) = a_3(\beta)'$

Since $a_3(\alpha) = ga_3(\beta) = a_3(\alpha)' = g'a_3(\beta)'$, $g = g'$ from (29). This is a contradiction to the assumption on A_3 and A_3'. Thus, x can't be supported as a FE allocation relative to $\{A_h'\}$ and this is true for any arbitrary FE relative to $\{A_h\}$. This means that the set of FE allocations are generically disjoint each other relative to a very small change in financial opportunities. This shows another aspect of the volatility of financial economy.

V. CONCLUDING REMARK

It has been shown that contingent securities model with restrictions on the portfolio choices can yield a real indeterminacy of equilibria as the canonical model of incomplete financial markets. Thus, contingent securities model is rich enough to discuss all the problems possibly arising in economies with incomplete financial markets, including those problems associated with restricted participations in financial markets.[10] For any restrictions on contingent securities choices considered here can be reinterpreted as a certain type of restricted participations in financial markets where returns from assets are in a general position.

The basic message here is that economies with financial assets are quite indeterminate so that any little perturbation of some parameters in the economy will yield a continuum of equilibrium allocations. As is known, there appears real indeterminacy of equilibria with a dimensional property even if a single asset is missing from a complete financial markets. Furthermore, the same phenomenon can also prevail even if financial markets are complete, but there is just a single consumer who is minimally restricted in choosing his portfolio. This shows the tremendous lack of robustness of complete market hypothesis with respect to a small perturbation of parameters in economies with financial markets. By imposing various types of restrictions on contingent securities model, we can not only examine the effect of such perturbations on the set of equilibrium allocations, but also investigate other problems such as the existence of Pareto optimal FE.

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