THE KOREAN ECONOMIC REVIEW
Volume 11, Number 2, February 1996.

A TWO SECTOR INTERACTION MODEL WITH INTRA- AND INTER- SECTORAL TECHNOLOGY SPILLOVERS*

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This paper presents a sectoral interaction model of business cylces with intra- and inter- sectoral technology spillovers. Given microeconomic decision rules, aggregate dynamics is derived using mean-field approximation. Technological progress is partially endogenized through interactive process in contrast to the real business cycle model which assumed fully exogenous shocks. Depending on parameter values, the economy exhibits multiple steady states, in which case uncorrelated sector specific shocks cause aggregate fluctuations through inter-sectoral shock mechanism.

I. INTRODUCTION

This paper presents a dynamic stochastic model of business cycles in a two sector economy. The interaction among a large number of firms which adopt particular technologies in the presence of technological externalities is analyzed.

In contrast to one sector interaction model in which the interaction pattern and magnitude of interaction across all firms were assumed to be symmetric and identical, by allowing for differences in the strength of intra- versus inter- industry externalities, the two sector model closely resembles the real economy which is characterized by multiple, diverse sectors. For example, telecommunications, transportations, computers and information service sectors create large, positive externalities to other sectors by reducing transaction costs and data processing and monitoring costs. In addition, knowledge spillovers emanating from competing firms in high technology sectors should have a greater impact on firms in other electronics industries than, say, the agricultural sector. By presenting a simple model

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* This is a revised version of the paper presented at the FIAC Workshop, Amsterdam, Netherlands, 1994. It was also presented at the annual meeting of the Korean Economic Association, Seoul, Korea. I would like to thank the commentator Yongyeob Son for his helpful comments.

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of intra- and inter- sectoral interactions, a richer set of aggregate dynamic features can be captured, including the number of stochastic steady states, the stability of these steady states, and the dynamic adjustment process.

The literature regarding sectoral interactions, which will not be summarized here, includes the following: Murphy, et al. (1989) who examine the interaction between the agricultural and manufacturing sectors; Long and Plosser (1983) who introduce sector specific shocks into a real business cycle model; Howitt (1985) who incorporates transaction costs between the labor and output markets; and Caballero and Lyons (1991) who conduct empirical tests to measure the size of externality effects in the manufacturing sector. Most of the above models are static ones. Some of them are dynamic but focus on the single steady state. In our model, there can exist multiple steady states and the explicit form of the aggregate transitional dynamics is derived.

Firms face idiosyncratic cost shocks and make decisions asyncronously in the continuous time. Specifying microeconomic decision rules, a probability flow model for aggregate dynamics of the economy is derived, and solved using mean-field approximation.[1] This model combines aspects found in real business cycle models such as technology shocks, with features found in Keynesian models such as coordination failure. It partially endogenizes technological progress in terms of interactive process, in contrast to the real business cycle model which assumed fully exogenous shocks.

Depending on parameter values such as spillover magnitudes and firm-specific cost shock variance, the economy exhibits multiple stochastic steady states. In this case, small and uncorrelated sector specific shocks cause aggregate fluctuations and co-movements among sectors' activity levels through inter-sectoral shock mechanism. Fluctuations in our model are due to the economy's movements (stochastic transition) across multiple steady states. This model has an implication that sector-specific shocks can have economy-wide impacts. In principle, this model can be extended to more than two sectors.

This paper is organized as follows: Section 2 presents the model, describing the economic environment, individual firm's decision rules, and aggregate dynamics. The main qualitative results for this paper are summarized graphically in Section 3 which presents simulation results. Section 4 concludes the paper.

II. MODEL

2.1. Economic Environment and Microeconomic Specification: Individual Firm's Decision Rule

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[1] It denotes a stochastic approximation method, originally used in physics, which replaces complex interaction terms with an average interaction term (see Weidlich and Haag (1983) and Ellis (1985)).

The economy is composed of two production sectors or industries. There are N_1 firms in sector 1 and N_2 firms in sector 2, totalling to N. All firms are monopolists producing their own perishable goods. Each firm faces the same downward sloping market demand curve. The specific form is the following:

$$P_t = a\,Y_t^b \tag{1}$$

Where P_t represents price, Y_t output, and t denotes time. Assume that $a > 0$ and $0 < b < 1$. In particular, $1/b$ the elasticity of demand, is assumed to be greater than one.

Firms in each sector have choices between two production techniques, a non-innovative technology which generates no externalities and an innovative technology which raises other firms' productivity through technology spillovers by a global diffusion process, for example, through direct input linkages or indirect knowledge spillovers. This improved production method requires the firm to bear innovation costs.

By technological spillovers, we mean that originators of the technology cannot fully appropriate their contributions. This public good aspect (partial non-excludability) of technology leads to the discrepancy between social and private returns to invention and innovations. It follows from the abstract, complex, and intangible nature of technology. Legal protection of property right for technology is difficult to enforce perfectly due to monitoring and information processing costs, imperfect patent laws, and imitation both in domestic and international markets. In addition, purchasers with market power often get extra benefits by buying the advanced materials from input suppliers (pecuniary technological externality). As social communication and transportation system develop, national education level rises, and the number of research institutions increases, the degree of these positive externalities will increase.[2]

The production techniques for sector 1 and 2 are described as follows.

[Sector 1]

Technology 1 (innovative technology) :

$$Y_{1,t}^{1} = \gamma_1(x_{1,t}, x_{2,t})\,L_{1,t}$$

Technology 2 (non-innovative technology) :

$$Y_{1,t}^{2} = L_{1,t}$$

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[2] Grossman and Helpman (1991) discuss in more detail several mechanisms for the propagation of technological spillovers such as public knowledge capital, investigation of new products, mobility of skilled labor among firms, and published research findings. Also there is a substantial empirical literature on technological externalities (Mansfields et al., Bernstein and Nadiri (1988, 1989)).

[Sector 2]
Technology 1 (innovative technology) :
$Y_{2,t}^{1} = \gamma_2(x_{1,t}, x_{2,t}) L_{2,t}$
Technology 2 (non-innovative technology) :
$$Y_{2,t}^{2} = L_2 \tag{2}$$

where $x_{1,t} = n_{1,t}/N_1$ and $x_{2,t} = n_{2,t}/N_2$

$n_{1,t}$ = number of firms adopting technology 1 in sector 1
$n_{2,t}$ = number of firms adopting technology 1 in sector 2
$x_{1,t}$ = fraction of firms adopting technology 1 in sector 1
$x_{2,t}$ = fraction of firms adopting technology 1 in sector 2
$0 \leq x_{1,t} \leq 1$ and $0 \leq x_{2,t} \leq 1$
$L_{i,t}$=labor input in sector i, i = 1, 2
$\gamma_i(.,.)$=measure of technological externalities in sector i, and $\gamma_i \geq 1$, i=1, 2

where superscripts denote technologies, subscripts denote sectors and t denotes time.

In both technologies, labor is the only primary input. Assume that the labor market is perfectly competitive with wage rate normalized to one in both sectors.

Idiosyncratic shocks change the production cost only for technology 1 which makes the model stochastic.[3] Since the wage rate is one under both technologies and technology 1 incurs innovation costs, the production costs for two technologies become:

$$C_{i,t}^{1} = L_{i,t} + F_{i,t} + e_{i,t},$$
$$C_{i,t}^{2} = L_{i,t} \tag{3}$$

Where $F_{i,t}$ indicates the innovation cost common to all firms in sector i for i = 1, 2. $e_{i,t}$ represents idiosyncratic cost shocks to individual firms in sector i (i = 1, 2) and is assumed to follow Normal $(0, \sigma^2)$.

From equations (1), (2), and (3), the profits for both technology adoptions have the following expressions:

$$\Pi_{i,t}^{1} = a[\gamma_i(x_{1,t}, x_{2,t})]^{\alpha} L_{i,t}^{\alpha} - L_{i,t} - F_{i,t} - e_{i,t}$$
$$\Pi_{i,t}^{2} = a L_{i,t}^{\alpha} - L_{i,t}$$

where $\Pi_{i,t}^{1}$, $\Pi_{i,t}^{2}$ denote profits generated by technology 1 and technology 2 in sector i respectively. $\alpha=1-b$ and $0 < \alpha < 1$ since $0 < b < 1$.

From First Order Conditions, the maximized profits for both technology

[3] We can easily analyze a model with uncertainty in both technology choices.

adoptions become:

[Sector 1]

$$\Pi_{1,t}^{1} = (a\alpha)^{(1/1-\alpha)}(\alpha^{1}-1)[\gamma_1(x_{1,t}, x_{2,t})]^{(\alpha/1-\alpha)} - F_{1,t} - e_{1,t}$$
$$\Pi_{1,t}^{2} = (a\alpha)^{(1/1-\alpha)}(\alpha^{1}-1) \qquad (4)$$

[Sector 2]

$$\Pi_{2,t}^{1} = (a\alpha)^{(1/1-\alpha)}(\alpha^{1}-1)[\gamma_2(x_{1,t}, x_{2,t})]^{(\alpha/1-\alpha)} - F_{2,t} - e_{2,t}$$
$$\Pi_{2,t}^{2} = (a\alpha)^{(1/1-\alpha)}(\alpha^{1}-1) \qquad (5)$$

When individual firm makes a decision, it observes the current aggregate technology state, represented by $(x_{1,t}, x_{2,t})$, and its own idiosyncratic cost shocks, represented by $e_{i,t}$, prior to making an adoption decision. After the realization of a cost shock, the firm compares the realizable profits from both technology adoptions and chooses the one yielding a higher profit. This makes each firm's technology choices random. The individual firm's decision rule is represented by the following technology choice probability:

$$\rho_1(x_{1,t}, x_{2,t}) = \text{Prob}(\Pi_{1,t}^{1} \geq \Pi_{1,t}^{2}) = \text{SCNDF}(Z_{1,t}/\sigma)$$
$$\rho_2(x_{1,t}, x_{2,t}) = \text{Prob}(\Pi_{2,t}^{1} \geq \Pi_{2,t}^{2}) = \text{SCNDF}(Z_{2,t}/\sigma) \qquad (6)$$

where $\rho_i(.,.)$ is the probability of choosing technology 1 for firms in sector i at time t for i = 1, 2, and SCNDF(.) denotes the standard cumulative normal distribution function. From equations (4) and (5), the variable $Z_{i,t}$ is represented as:

$$Z_{i,t} = (a\alpha)^{(1/1-\alpha)}(\alpha^{1}-1)[(\gamma_i(x_{1,t}, x_{2,t}))^{(\alpha/1-\alpha)} - 1] - F_{i,t} \quad i = 1, 2 \qquad (7)$$

The externality measures are specified as follows:

$$\gamma_1(x_{1,t}, x_{2,t}) = 1 + a_{11}x_{1,t} + a_{12}x_{2,t}$$
$$\gamma_2(x_{1,t}, x_{2,t}) = 1 + a_{21}x_{1,t} + a_{22}x_{2,t} \qquad (8)$$

where a_{ij} is the parameter measuring the influence on productivity from sector j to sector i. For j=i, the spillover is intra-sectoral and inter-sectoral, otherwise. Spillover effects can thus be summarized in a two-by-two matrix.

For a concrete illustration, set a=2 and α=1/2 (the same values for these parameters are used to conduct simulations). Then from equations (6), (7) and (8), the technology choice probability becomes:

$$\rho_i(x_{1,t}, x_{2,t}) = \text{SCNDF}(((a_{i1}x_{1,t} + a_{i2}x_{2,t}) - F_{i,t})/\sigma), \quad i = 1, 2 \qquad (9)$$

The choice probability implies a simple Markovian structure in that the firm's technology state in the next instant depends on the current aggregate technology state, $(x_{1,t}, x_{2,t})$. This simple structure results from the assumption of myopic decision-making on the part of firms which is caused by the absence of dynamic linkages in the physical variables of this model.

As the fraction of firms using the innovative technology in both sectors increases, profits from adopting the innovative technology increases as long as $a_{ij} > 0$. Therefore, the probability of choosing the innovative technology in both sector increases. Depending on the values of the a_{ij}, aggregate dynamics exhibit several patterns due to multi-sector interactions. The magnitude of a_{ij} reflects spillover propensity, which is influenced by numerous factors, including heterogeneity of production processes in an industry and communication systems, defined broadly.

The normalized output, y_t, which is equal to total output (GNP) divided by total number of firms, N, will be determined by the following relationships:

$$y_t = \mathrm{f} y_{1,t} + (1-\mathrm{f}) y_{2,t}$$
$$y_{i,t} = x_{i,t} Y_{i,t}^{1} + (1-x_{i,t}) Y_{i,t}^{2}, \quad i=1,2 \tag{10}$$

where $y_{i,t}$ denotes the normalized output of sector i, i=1,2, and $\mathrm{f}=N_1/N$ denotes the fraction of firms in sector 1.

The normalized sectoral output, equal to sectoral output divided by the number of firms in that sector can be derived. In our specific example where a=2 and $\alpha=1/2$, it becomes:

$$y_{i,t} = x_{i,t}(1 + a_{i1}x_{1,t} + a_{i2}x_{2,t})^2 + (1-x_{i,t}), \quad i=1,2 \tag{11}$$

2.2. Aggregate Dynamics

In this section, the aggregate dynamics is derived to describe the evolution of the fraction of firms adopting the innovative technology in both sectors given the individual firm's decision rule specified in the previous section.

The dynamic linkage between individual technology choice and aggregate technology state is the following; Individual firm's current technology choice which depends on the current aggregate technology state $(x_{1,t}, x_{2,t})$ determines the next instant aggregate technology state. In turn, the next instant aggregate technology state affects the next instant individual firm's decision. Through this feedback mechanism, the economy evolves over time. In standard Markov chain models, transition probabilities are exogenously given. Our model has a distinct feature in that transition probabilities changes endogenously with the aggregate state of the economy.

2.2.1. General Aggregate Dynamics[4]

This subsection presents aggregate dynamics based on the general form of transition probabilities. In the next subsection we derive the aggregate dynamics under the specific functional form of individual transition probabilities represented in section 2.1. The material in this subsection is highly technical and we place it in the Appendix.

2.2.2. Specification of Aggregate Dynamics

This section applies the general formulation derived in the Appendix to the economic environment described in section 2.1.

In our model, the individual transition probability from technology i to technology j in sector s (denoted by v_{ji}^{s}, i, j, s=1,2) is independent of the current individual state but dependent on current aggregate state of the economy. Individual transition probabilities can be denoted by the following structure:

$$v_{12}^{1} = \rho_1(x_1, x_2) \qquad v_{21}^{1} = 1 - \rho_1(x_1, x_2)$$
$$v_{12}^{2} = \rho_2(x_1, x_2) \qquad v_{21}^{2} = 1 - \rho_2(x_1, x_2)$$

From equation (7) of Appendix, the corresponding aggregate transition probability becomes:

$$\begin{aligned} w[-1, 0 ; n_1, n_2] &= n_1(1-\rho_1(x_1, x_2)) \\ w[+1, 0 ; n_1, n_2] &= (N_1-n_1)\rho_1(x_1, x_2) \\ w[0, -1 ; n_1, n_2] &= n_2(1-\rho_2(x_1, x_2)) \\ w[0, +1 ; n_1, n_2] &= (N_2-n_2)\rho_2(x_1, x_2) \end{aligned} \tag{12}$$

From equations (11) and (14) of Appendix and equation (12):

$$\begin{aligned} K_1(x_1, x_2) &= \rho_1(x_1, x_2) - x_1 \\ K_2(x_1, x_2) &= \rho_2(x_1, x_2) - x_2 \\ Q_1(x_1, x_2) &= (1-2x_1)\rho_1(x_1, x_2) + x_1 \\ Q_2(x_1, x_2) &= (1-2x_2)\rho_2(x_1, x_2) + x_2 \end{aligned} \tag{13}$$

From equations (15) and (16) of Appendix and equation (13), the basic equations for the mean and the variance-covariances of the aggregate state of the economy are obtained. The dynamics for the mean are described by the following two equations:

[4] This section is based on the framework in Weidlich and Haag (1983 and 1988).

$$
\begin{aligned}
dx_1/dt &= \rho_1(x_1(t),\ x_2(t)) - x_1(t) \\
dx_2/dt &= \rho_2(x_1(t),\ x_2(t)) - x_2(t)
\end{aligned} \tag{14}
$$

The dynamics for the variance-covariance are given by:

$$
\begin{aligned}
d\sigma(x_1)/dt &= ((1-2x_1)\rho_1 + x_1)/N_1 + 2\sigma(x_1)((\partial\rho_1/\partial x_1) - 1) \\
&\quad + 2\sigma(x_1x_2)(\partial\rho_1/\partial x_2) \\
d\sigma(x_1x_2)/dt &= \sigma(x_1)(\partial\rho_2/\partial x_1) + \sigma(x_1x_2)[(\partial\rho_1/\partial x_1) \\
&\quad + ((\partial\rho_2/\partial x_2) - 1)] + \sigma(x_2)(\partial\rho_1/\partial x_2) \\
d\sigma(x_2)/dt &= ((1-2x_2)\rho_2 + x_2)/N_2 \\
&\quad + 2\sigma(x_2)((\partial\rho_2/\partial x_2) - 1) + 2\sigma(x_1x_2)(\partial\rho_2/\partial x_1)
\end{aligned} \tag{15}
$$

where $\rho_i \equiv \rho_i(x_1(t),\ x_2(t)),\quad i = 1,\ 2$

From equation (14), steady states of the aggregate state of the economy are determined by the solutions to the following simultaneous equations:

$$
\begin{aligned}
dx_1/dt &= \rho_1(x_1(t),\ x_2(t)) - x_1(t) = 0 \\
dx_2/dt &= \rho_2(x_1(t),\ x_2(t)) - x_2(t) = 0
\end{aligned} \tag{16}
$$

Equivalently,

$$
\begin{aligned}
x_1 &= \rho_1(x_1,\ x_2) \\
x_2 &= \rho_2(x_1,\ x_2)
\end{aligned}
$$

The number of steady states depends on structural parameter values such as the interaction parameters a_{ij}.

To obtain some insights into the global features of aggregate dynamics, first perform a linear stability analysis around the steady states. Let $h_1(t) = x_1(t) - x_1$ and $h_2(t) = x_2(t) - x_2$ where $(x_1,\ x_2)$ represent steady states.

Linearization of equation (14) in $h_1(t)$, $h_2(t)$ gives:

$$
\begin{aligned}
dh_1(t)/dt &= (r_{11} - 1)h_1(t) + r_{12}h_2(t) \\
dh_2(t)/dt &= (r_{21} - 1)h_1(t) + r_{22}h_2(t)
\end{aligned} \tag{17}
$$

where $r_{ij} = \partial\rho_i/\partial x_j \quad i,\ j = 1,\ 2.$

The solutions to equation (17) are linear combinations of the eigensolutions:

$$h_1(t)_\pm = h_1(0)\exp(\lambda_\pm t) \text{ and } h_2(t)_\pm = h_2(0)\exp(\lambda_\pm t)$$

where the eigenvalues are:

$$\lambda_{\pm}=(1/2)\{(r_{11}+r_{22}-2)\pm[(r_{11}+r_{22}-2)^2+4(r_{12}r_{21}-(r_{11}-1)(r_{22}-1))]^{1/2}\}$$

The eigenvalues are either real or conjugate complex numbers. The steady state is stable if the real parts of both eigenvalues are negative, while the steady state is unstable if the real part of one of the eigenvalues is positive. If the eigenvalues are real (complex), $x(t)\equiv(x_1(t),\ x_2(t))$ approaches or moves away from the steady state linearly (spirally).

The stationary distribution $P_s(n_1,\ n_2)$ can be derived from the Master Equation (8) of Appendix with the following probability normalization condition:

$$\sum_{n_1}\sum_{n_2}P_s(n_1,\ n_2)=1.$$

Ⅲ. SIMULATION OF THE MODEL

This section presents the simulation results of equation (14) with the technology choice probability specified as in equation (9). Figures [1] through [12] display graphically the simulation results based on alternative scenarios about spillover magnitudes and firm-specific cost shock variance. Throughout, a shock is introduced only to one sector.

Figure [1] shows the effects of a shock to sector 1 under the assumption of intra-industry spillovers but no inter-industry spillovers. In this case, the 2-by-2 matrix measuring externalities is diagonal reflecting the absence of inter-sectoral spillovers. Figure [1] establishes a baseline case comparable to the one sector model. As the graph shows, sector 2 adoption rates remain unchanged during the entire simulation.

In Figure [2], the shock perturbs sector 1 and spills over into sector 2. The externality matrix has all elements equal to 1. As the figure shows, the trend in adoption rates of sector 2 follows sector 1, though sector 1, where the shock originates, exhibits greater volatility. Figure [2] demonstrates the case of multiple steady states. Figure [3] shows the same situation as Figure [2] except that only a single steady state is imposed on the model. As such, Figure [3] shows greater mean reversion around an average adoption rate roughly equal to 0.5.

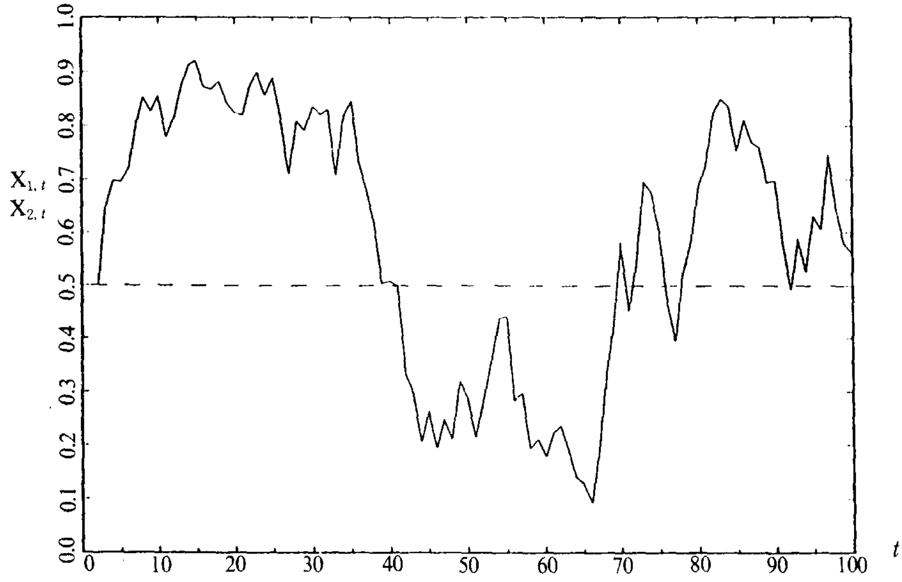
From equations (10) and (11) which represent the relation between technology adoption rates and outputs, evolution of outputs can be simulated. Figure [4] and Figure [5] show dynamics of normalized sectoral and total outputs ($y_{i,t}$, $i=1$, 2 and y_t) with the same parameter values as in Figure [2]. Similarly Figure [6] and Figure [7] show simulation results with the same parameter values as in Figure [3]. The difference between simulations with multiple versus single steady states are evident by the former's dramatic changes in sectoral and aggregate outputs, which are caused by the aggregate transition between low and high steady states through inter-sectoral shock mechanism.

Figure [8] shows strong intra-sectoral interactions, relative to inter-sector spillovers. The shock originates in sector 1 which shows high volatility and spills over to sector 2. The trend in sector 2 follows sector 1 but is much smoother due to less inter-sectoral interaction. For comparison, Figure [9] shows the opposite case: high inter-sectoral interaction with low intra-sectoral interaction. In this case, because spillovers are pronounced, sector 2 (the receiving sector) follows sector 1 very closely both in terms of trend and volatility.

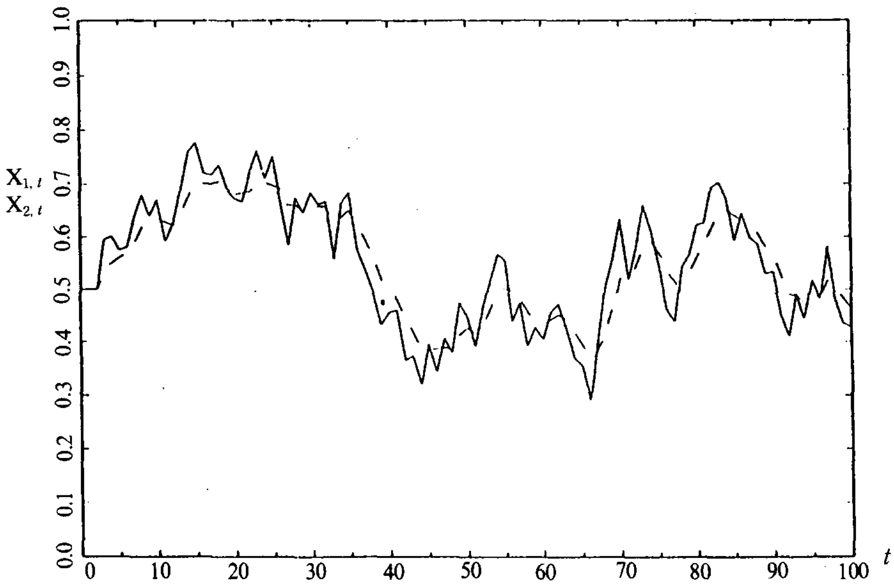
Figures [10] and [11] show scenarios where the shock perturbs the "leading" and "following" sectors, respectively. The leading sector exerts high inter-sectoral spillover effects. Thus, shocks originating in this sector have a strong impact on downstream sectors. The "following" sector exhibits very little inter-sectoral externality effects and is the more interesting case. Shocks to "follower" firms lead to high volatility in this sector but have only a sluggish effect on the leading sector, whose smooth trend parallels the "follower" sector.

Finally, Figure [12] shows the case of a limit cycle. A congestion scenario was constructed by making one of the offdiagonal elements of externality matrix negative. In this case, sector 2 was assumed to exert a negative externality effect on sector 1's production process.

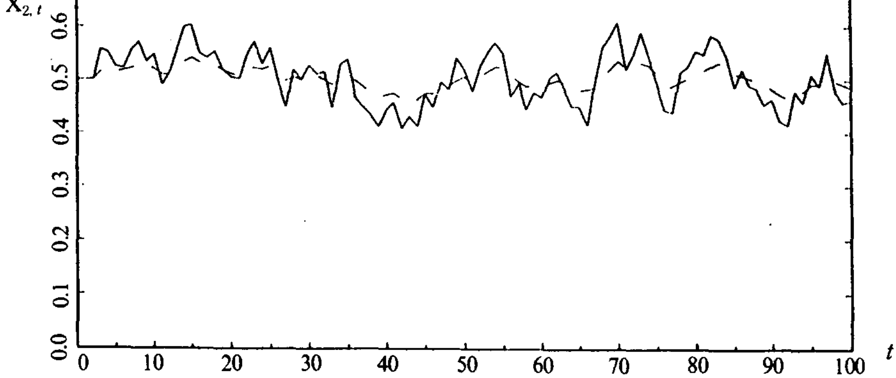
[Figure 1] Intra-Sector Interaction Only. The Solid line denotes the evolution of $x_{1,t}$ and the dotted line donotes the evolution of $x_{2,t}$. $a_{11} = a_{22} = 2$, $a_{12} = a_{21} = 0$ and $\sigma = 0.7$. $F_{1,1} = F_{2,t} = 1$ and $F_{1,t} = F_{1,1} + u_t$, $u_t \sim$ Uni $[-0.3, 0.3]$, if $t \geq 2$.



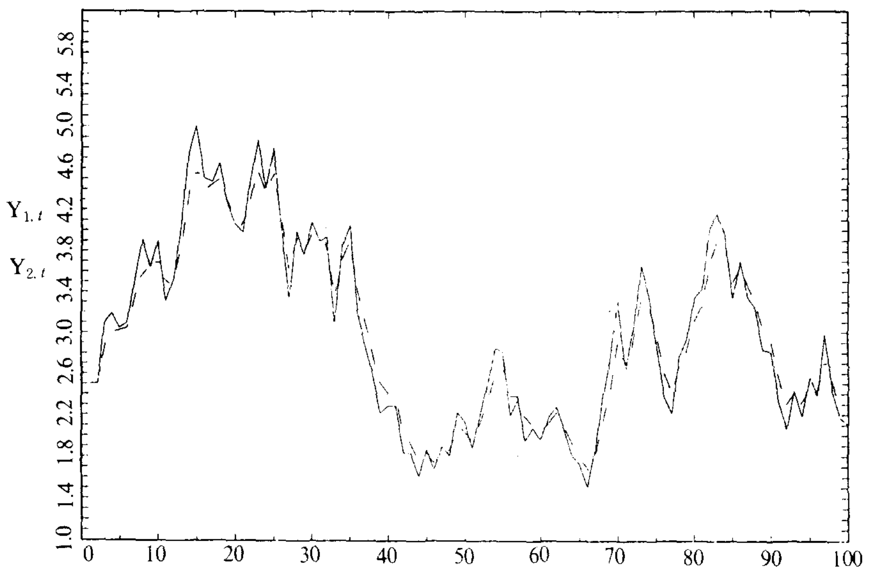
[Figure 2] Identical Intra- and Inter-Sector Interaction Under Multiple Steady States. $a_{11}=a_{22}=a_{12}=a_{21}=1$ and $\sigma=0.8$. $F_{1,1}=F_{2,t}=1$ and $F_{1,t} = F_{1,1}+u_t$, $u_t \sim \text{Uni}[-0.3, 0.3]$, if $t \geq 2$.



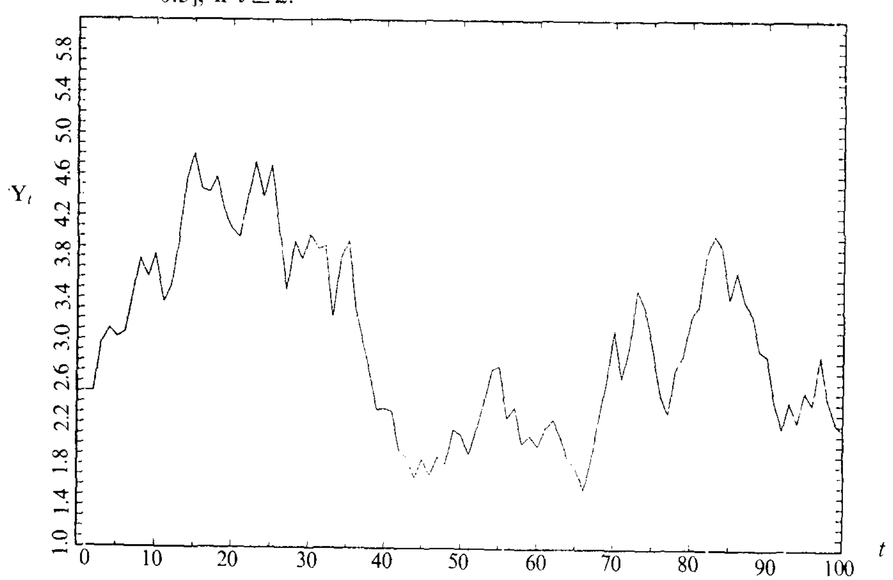
[Figure 3] Identical Intra- and Inter-Sector Interaction Under a Single Steady States. $a_{11}=a_{22}=a_{12}=a_{21}=1$ and $\sigma=1.2$. $F_{1,1}=F_{2,t}=1$ and $F_{1,t} = F_{1,1}+u_t$, $u_t \sim \text{Uni}[-0.3, 0.3]$, if $t \geq 2$.



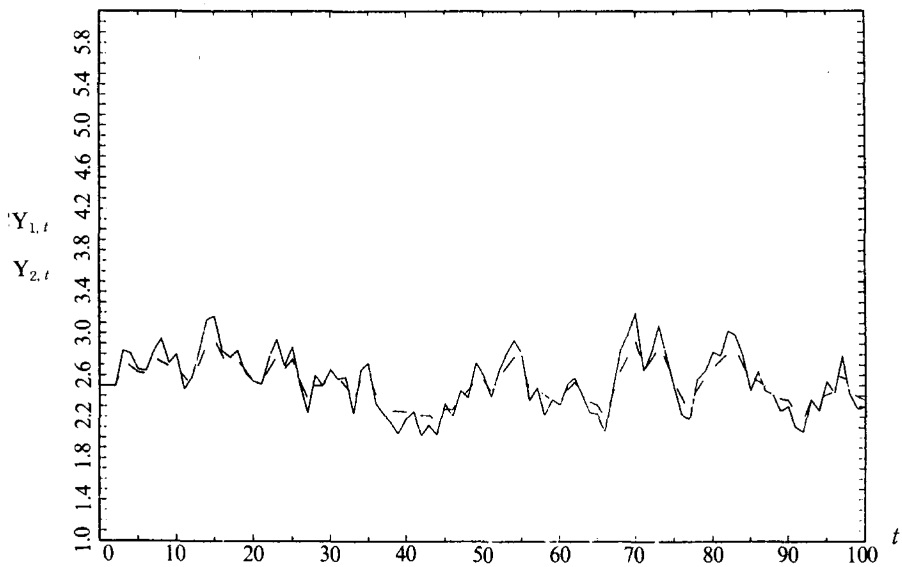
[Figure 4] Evolution of Normalized Sectoral Outputs Under Multiple Steady States. Identical Intra- and Inter-Sector Interaction. The solid line denotes the evolution of $y_{1,t}$ and the dotted line denotes the evolution of $y_{2,t}$. $a_{11}=a_{22}=a_{12}=a_{21}=1$ and $\sigma=0.8$. $F_{1,1}=F_{2,t}=1$ and $F_{1,t}=F_{1,1}+u_t$, $u_t\sim$ Uni[-0.3, 0.3], if $t\geq 2$.



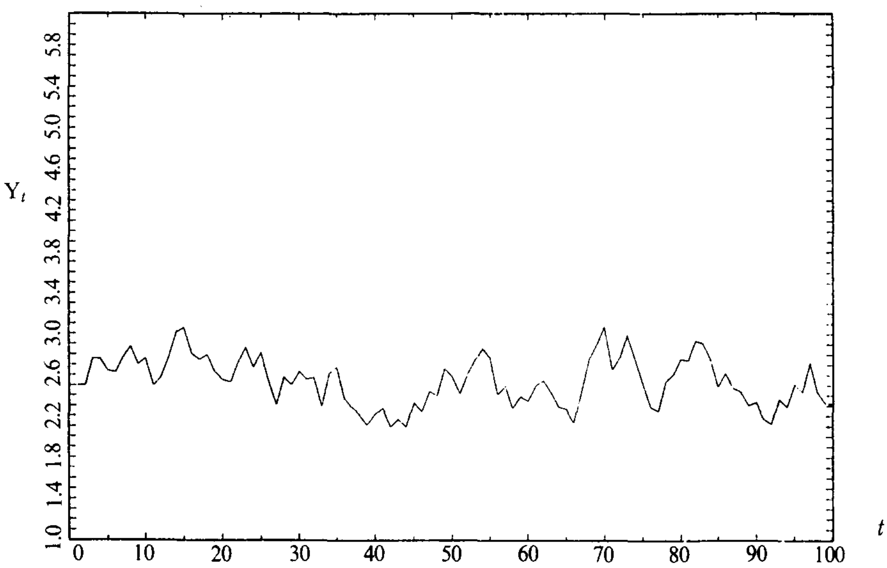
[Figure 5] Evolution of Normalized Total Output Under Multiple Steady States. Identical Intra- and Inter-Sector Interaction. $a_{11}=a_{22}=a_{12}=a_{21}=1$ and $\sigma=0.8$. $F_{1,1}=F_{2,t}=1$ and $F_{1,t}=F_{1,1}+u_t$, $u_t\sim$ Uni[-0.3, 0.3], if $t\geq 2$.



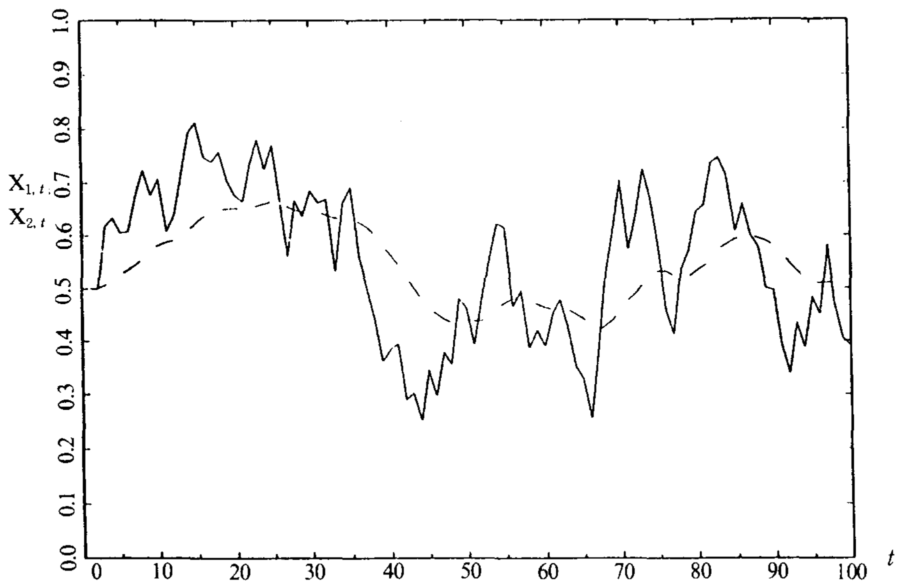
[Figure 6] Evolution of Normalized Sectoral Outputs Under a Single Steady States. Identical Intra- and Inter-Sector Interaction. $a_{11}=a_{22}=a_{12}=a_{21}=1$ and $\sigma=1.2$. $F_{1,1}=F_{2,t}=1$ and $F_{1,t}=F_{1,1}+u_t$, $u_t \sim$ Uni[−0.3, 0.3], if $t \geq 2$.



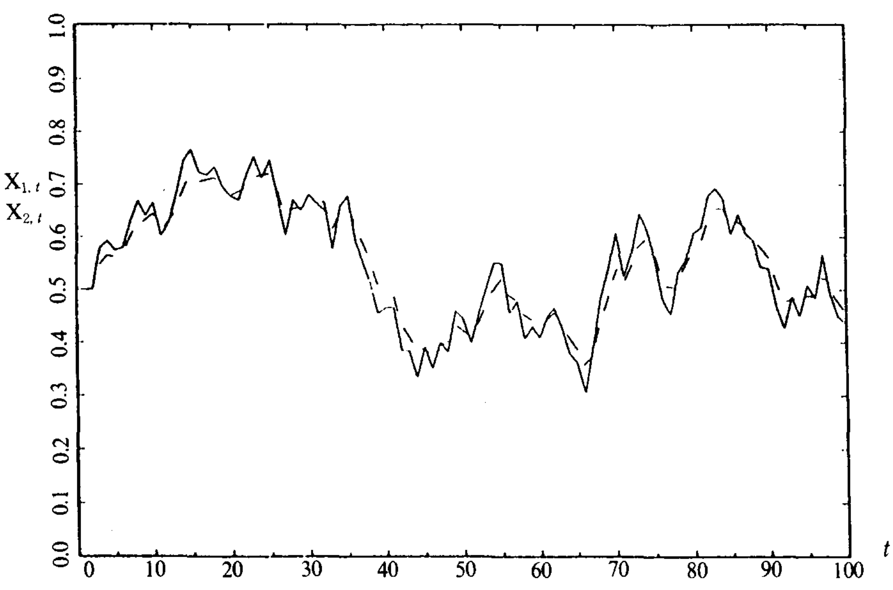
[Figure 7] Evolution of Normalized Total Output Under a Single Steady State. Identical Intra- and Inter-Sector Interaction. $a_{11}=a_{22}=a_{12}=a_{21}=1$ and $\sigma=1.2$. $F_{1,1}=F_{2,t}=1$ and $F_{1,t}=F_{1,1}+u_t$, $u_t \sim$ Uni[−0.3, 0.3], if $t \geq 2$.



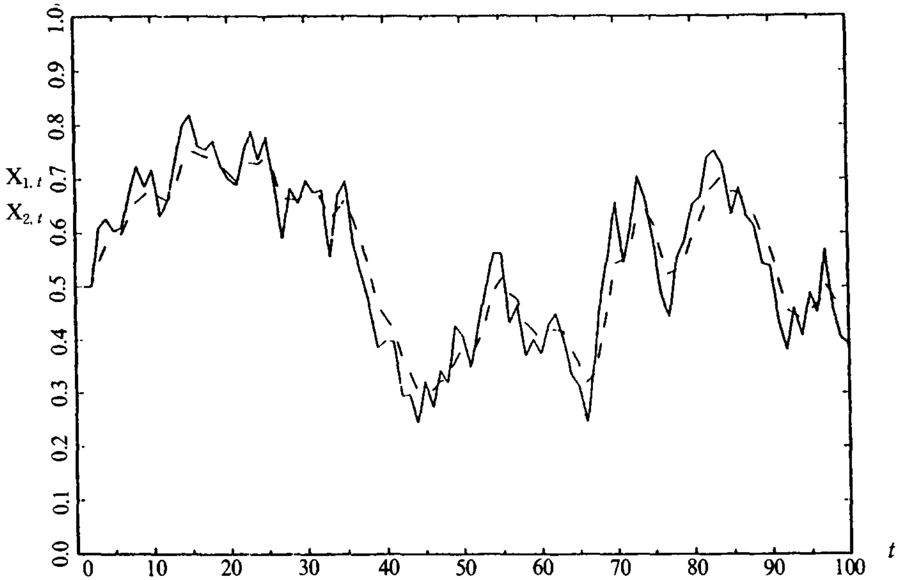
[Figure 8] Strong Intra-Sector Interaction. $a_{11}=a_{22}=1.8$, $a_{12}=a_{21}=0.2$ and $\sigma=0.8$. $F_{1,1}=F_{2,t}=1$ and $F_{1,t}=F_{1,1}+u_t$, $u_t \sim \mathrm{Uni}[-0.3, 0.3]$, if $t \geq 2$.



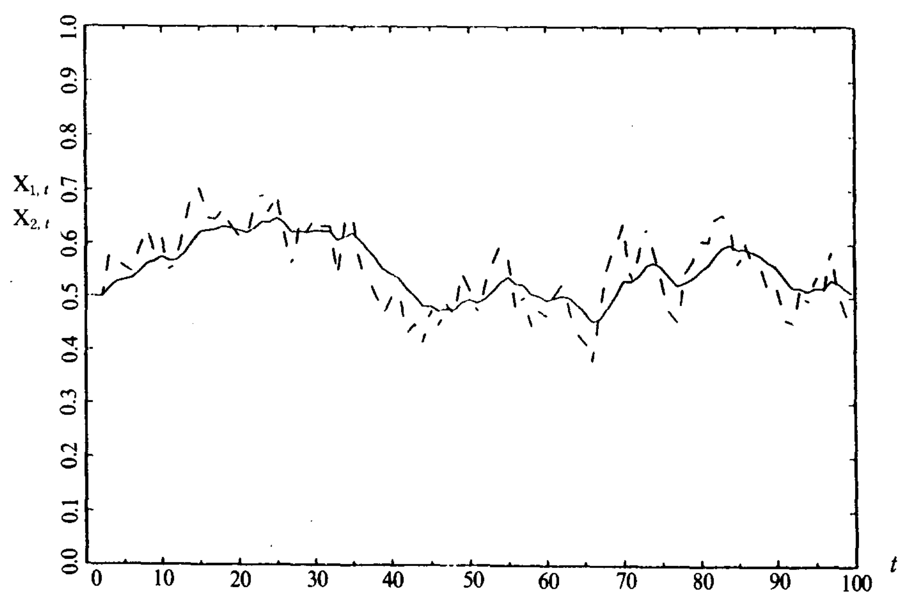
[Figure 9] Strong Inter-Sector Interaction. $a_{11}=a_{22}=0.5$, $a_{12}=a_{21}=1.5$ and $\sigma=0.8$. $F_{1,1}=F_{2,t}=1$ and $F_{1,t}=F_{1,1}+u_t$, $u_t \sim \mathrm{Uni}[-0.3, 0.3]$, if $t \geq 2$.



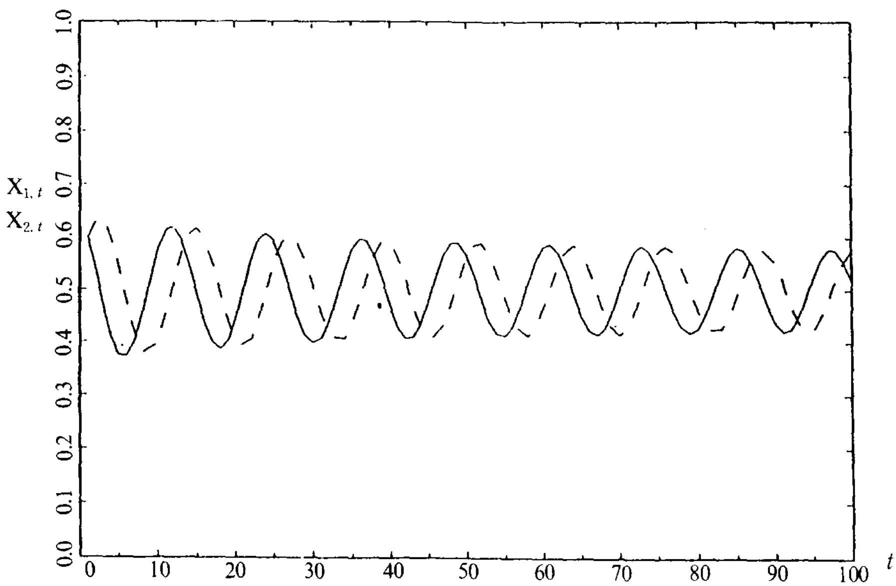
[Figure 10] Shocks to Leading Sector. $a_{11}=a_{21}=1.5$, $a_{22}=a_{12}=0.5$ and $\sigma=0.8$. $F_{1,1}=F_{2,t}=1$ and $F_{1,t}=F_{1,1}+u_t$, $u_t \sim$ Uni[−0.3, 0.3], if $t \geq 2$.



[Figure 11] Shocks to Non-Leading Sector. $a_{11}=a_{21}=1.5$, $a_{22}=a_{12}=0.5$ and $\sigma=0.8$. $F_{1,t}=F_{2,1}=1$ and $F_{2,t}=F_{2,1}+u_t$, $u_t \sim$ Uni[−0.3, 0.3], if $t \geq 2$.



[Figure 12] Limit Cycle Under Congestion. $a_{11}=a_{22}=1$, $a_{12}=-0.5$, $a_{21}=0.5$ and $\sigma=0.4$. $F_{1,t}=0.25$ and $F_{2,t}=0.75$, $t\geq 1$. $x_{1,1}=x_{2,1}=0.6$



Ⅳ. CONCLUSION AND LIMITATIONS

This paper introduces a multi-dimensional birth-and-death stochastic process with endogenized transition rates as an alternative modelling framework for aggregate behavior of a large number of interacting agents. We apply it to a two sector interaction model of business cycles with intra- and inter- sectoral technological externalities.

By specifying microeconomic decision rules, the time evolution of the average fractions of firms adopting the innovative technology in both sectors are derived using mean-field approximation. In this model, technological progress is endogenized through interactive process in contrast to the real business cycle model which assumed fully exogenous serially correlated shocks. Aggregate fluctuations and co-movements among sectors' activity levels were generated through an inter-sectoral shock mechanism, where shocks perturbed only a single sector. Hence, this model has the implication that sector-specific shocks can have economy-wide impacts.

This model represents a substantial improvement over one sector model. The two sector model more closely resembles the real economy which is characterized by complex and possibly heterogeneous interactions among multiple, diverse industries.

A more complex model can be designed which considers separate shocks to both sectors. In addition, empirical estimates of the 2-by-2 (or of larger dimension in the case of more sectors) should be supplied to improve the applicability of the model. In principle, the model of this paper could be extended to multiple sectors.

APPENDIX

Aggregate dynamics of the model can be derived from the general Master Equation (a differential equation in time for the probability distribution) by an approximation. Under a Markov assumption, the following relationship between probability distributions holds:

$$P(j, t+\Delta t) = \sum_{(i)} P(j, t+\Delta t \,|\, i, t)\, P(i, t) \tag{A.1}$$

where $i, j \in$ I, I is a discrete state space, $P(i, t)$ is the probability of state i at time t, and $P(i, t+\Delta t \,|\, j, t)$ is the conditional probability of state i at time $t+\Delta t$ given state j at time t.

A first-order Taylor expansion of the conditional probability around t with respect to the variable $t_2 = t + \Delta t$ yields:

$$P(j, t+\Delta t \,|\, i, t) = P(j, t \,|\, i, t) + \Delta t \partial P(j, t_2 \,|\, i, t)/\partial t_2 \,|_t + 0(\Delta t^2) \tag{A.2}$$

By the definition of conditional probability,

$$\begin{aligned} &P(j, t \,|\, i, t) = 0 \text{ if } j \neq i \\ &P(j, t \,|\, i, t) = 1 \text{ if } j = i \end{aligned} \tag{A.3}$$

Since $\sum_{(j)} P(j, t_2 \,|\, i, t)=1$, we conclude that

$$\sum_{(j)} P(j, t_2 \,|\, i, t)/\partial t_2 \,|_t=0. \tag{A.4}$$

Substituting equations (A.3) and (A.4) into equation (A.2) gives

$$\begin{aligned} P(j, t+\Delta t \,|\, i, t) &= \Delta t w_t(j \,|\, i) + 0(\Delta t^2) \text{ if } j \neq i, \\ &= 1 - \Delta t \textstyle\sum_{j\neq i} w_t(j \,|\, i) + 0(\Delta t^2) \text{ if } j=i. \end{aligned} \tag{A.5}$$

where $w_t = \partial P(j, t_2 \,|\, i, t)/\partial t_2 \,|_t$ is the probability transition rate. Substituting equation (A.5) into equation (A.4) and taking limits yields the Master Equation:

$$\lim \Delta t \rightarrow 0 \, [P(j, t+\Delta t) - P(j, t)]/\Delta t = dP(j, t)/dt =$$

$$\sum_{i \neq j} w_t(j \mid i) P(i, t) - \sum_{i \neq j} w_t(i \mid j) P(j, t) \tag{A.6}$$

The left-hand-side (l.h.s.) of equation (A.6) (the change per unit time of the probability of state j) is composed of two counteracting terms: the probability inflow from all other states i to state j (the first term of the right-hand-side (r.h.s.) of equation (A.6)) and the probability outflow from state j to all other states i (the second term of the l.h.s. of equation (A.6)). The net change of the two terms determines the increase or decrease per unit of time of the probability $P(j, t)$.

In this model, the aggregate state of the economy is described by (n_1, n_2), a type of two dimensional lattice (integer) space,

$$0 \leq n_1 \leq N_1, 0 \leq n_2 \leq N_2.$$

Define the transition probability for the aggregate state as $w[k, l; n_1, n_2]$, which represents the transition probability from aggregate state (n_1, n_2) to $(n_1 + k, n_2 + l)$. For simplicity, assume that only one firm of each sector moves randomly in a short unit time interval. Then the following simple relationship between individual and aggregate transition probabilities holds:

$$\begin{aligned}
w[-1, 0; n_1, n_2] &= n_1 v_{21}^1(n_1, n_2) \\
w[+1, 0; n_1, n_2] &= (N_1 - n_1) v_{12}^1(n_1, n_2) \\
w[0, -1; n_1, n_2] &= n_2 v_{21}^2(n_1, n_2) \\
w[0, +1; n_1, n_2] &= (N_2 - n_2) v_{12}^2(n_1, n_2)
\end{aligned} \tag{A.7}$$

where v_{ji}^s is the individual firm's transition probability in sector s from technology i to technology j, $i, j, s=1, 2$.

The aggregate dynamics can now be derived as follows. From equation (A.6), we have,

$$\begin{aligned}
&dP(n_1, n_2; t) / dt = \\
&\quad w[-1, 0; n_1+1, n_2]\, P(n_1+1, n_2; t) - w[-1, 0; n_1, n_2]\, P(n_1, n_2; t) + \\
&\quad w[+1, 0; n_1-1, n_2]\, P(n_1-1, n_2; t) - w[+1, 0; n_1, n_2]\, P(n_1, n_2; t) + \\
&\quad w[0, -1; n_1, n_2+1]\, P(n_1, n_2+1; t) - w[0, -1; n_1, n_2]\, P(n_1, n_2; t) + \\
&\quad w[0, +1; n_1, n_2-1]\, P(n_1, n_2-1; t) - w[0, +1; n_1, n_2]\, P(n_1, n_2; t)
\end{aligned} \tag{A.8}$$

The mean value equations for n_1, n_2, n_1^2, $n_1 n_2$ and n_2^2 are derived from equation (A.8) using the following definition:

$$E(f(n_1, n_2)) = \sum_{n_1} \sum_{n_2} f(n_1, n_2) P(n_1, n_2; t)$$

Thus,

$$d\,E(f(n_1, n_2))/dt=\sum_{n_1}\sum_{n_2} f(n_1, n_2)dP(n_1, n_2;t)/dt,$$

where E(.) denotes expectation operator. The following boundary conditions also hold:

$$\begin{aligned}
w[-1, 0; 0, n_2] &= w[+1, 0; N_1, n_2]=0\\
w[0, -1; n_1, 0] &= w[0, +1; n_1, N_2]=0\\
P(N_1+1, n_2;t) &= P(n_1, N_2+1;t)=0.
\end{aligned}$$

The exact equations are:

$$\begin{aligned}
d\,E(n_1)_t/dt &= E(k_1(n_1, n_2))_t\\
d\,E(n_2)_t/dt &= E(k_2(n_1, n_2))_t
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
d\,E(n_1^2)_t/dt &= 2\,E(n_1k_1(n_1, n_2))_t + E(q_1(n_1, n_2))t\\
d\,E(n_1n_2)_t/dt &= E(n_1k_2(n_1, n_2))_t + E(n_2k_1(n_1, n_2))t\\
d\,E(n_2^2)_t/dt &= 2\,E(n_2k_2(n_1, n_2))_t + E(q_2(n_1, n_2))t
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
\text{where } k_1(n_1, n_2) &= w[+1, 0; n_1, n_2] - w[-1, 0; n_1, n_2]\\
k_2(n_1, n_2) &= w[0, +1; n_1, n_2] - w[0, -1; n_1, n_2]\\
q_1(n_1, n_2) &= w[+1, 0; n_1, n_2] + w[-1, 0; n_1, n_2]\\
q_2(n_1, n_2) &= w[0, +1; n_1, n_2] + w[0, -1; n_1, n_2]
\end{aligned} \tag{A.11}$$

Assuming a distribution $P(n_1, n_2;t)$ with only one sharp peak (uni-modal) around the mean values $E(n_1)_t \equiv n_{m\,1}(t)$ and $E(n_2)_t \equiv n_{m\,2}(t)$, approximate equations for equation (A.9) are obtained using a mean-field approximation where the expectation of a function is replaced with the function of an expectation:

$$\begin{aligned}
dn_{m\,1}/dt &= k_1(n_{m\,1}, n_{m\,2})\\
dn_{m\,2}/dt &= k_2(n_{m\,1}, n_{m\,2})
\end{aligned} \tag{A.12}$$

A first-order Taylor expansion of the r.h.s. of equation (A.10) around $(n_{m\,1}, n_{m\,2})$ yields the following approximate equations:

$$\begin{aligned}
d\sigma(n_1)/dt &= q_{m\,1}+2\sigma(n_1)(\partial k_{m\,1}/\partial n_1)+2\sigma(n_1n_2)(\partial k_{m\,1}/\partial n_2)\\
d\sigma(n_1n_2)/dt &= \sigma(n_1)(\partial k_{m\,2}/\partial n_1)+\sigma(n_1n_2)[(\partial k_{m\,2}/\partial n_2)+(\partial k_{m\,1}/\partial n_1)]\\
&\quad +\sigma(n_2)(\partial k_{m\,1}/\partial n_2)\\
d\sigma(n_2)/dt &= q_{m\,2}+2\sigma(n_2)(\partial k_{m\,2}/\partial n_2)+2\sigma(n_1n_2)(\partial k_{m\,2}/\partial n_1)
\end{aligned} \tag{A.13}$$

where $k_{m,i}=k_i(n_{m,1}, n_{m,2})$ and $q_{m,i}=q_i(n_{m,1}, n_{m,2})$.
Variances and covariance are denoted by:

$$\begin{aligned}
\sigma(n_1) &= E(n_1^2)_t - E(n_1)_t^2 \\
\sigma(n_2) &= E(n_2^2)_t - E(n_2)_t^2 \\
\sigma(n_1 n_2) &= E(n_1 n_2)_t - E(n_1)_t E(n_2)_t
\end{aligned}$$

For mathematical convenience[5], normalize n_1, n_2 with x_1, x_2:

$$x_1 = n_1 / N_1 \text{ and } x_2 = n_2 / N_2 \text{ where } 0 \leq x_1,\ x_2 \leq 1$$

Also let $k_1(n_1, n_2)=N_1\, K_1(x_1, x_2)$ and $k_2(n_1, n_2)=N_2\, K_2(x_1, x_2)$
$q_1(n_1, n_2)=N_1\, Q_1(x_1, x_2)$ and $q_2(n_1, n_2)=N_2\, Q_2(x_1, x_2)$ (A.14)

Normalized variances and covariance are denoted by

$$\begin{aligned}
\sigma(x_1) &= \sigma(n_1)/(N_1)^2 = E((x_1 - E(x_1))^2) \\
\sigma(x_1 x_2) &= \sigma(n_1 n_2)/(N_1 N_2) = E((x_1 - E(x_1))(x_2 - E(x_2))) \\
\sigma(x_2) &= \sigma(n_2)/(N_2)^2 = E((x_2 - E(x_2))^2)
\end{aligned}$$

Writing x_1, x_2 for $E(x_1)$, $E(x_2)$, the following mean value and variance- covariance equations hold

$$\begin{aligned}
dx_1 / dt &= K_1(x_1, x_2) \\
dx_2 / dt &= K_2(x_1, x_2)
\end{aligned} \tag{A.15}$$

$$\begin{aligned}
d\sigma(x_1) / dt &= Q_1/N_1 + 2\sigma(x_1)(\partial K_1/\partial x_1) + 2\sigma(x_1 x_2)(\partial K_1/\partial x_2) \\
d\sigma(x_1 x_2) / dt &= \sigma(x_1)(\partial K_2/\partial x_1) + \sigma(x_1 x_2)[(\partial K_1/\partial x_1) + \\
&\qquad (\partial K_2/\partial x_2)] + \sigma(x_2)(\partial K_1/\partial x_2) \\
d\sigma(x_2) / dt &= Q_2/N_2 + 2\sigma(x_2)(\partial K_2/\partial x_2) + 2\sigma(x_1 x_2)(\partial K_2/\partial x_1)
\end{aligned} \tag{A.16}$$

[5] See chap. 3 in Weidlich and Haag (1983) for exact derivation of mean values equations from Fokker-Plank Equation (continuum space version of Master Equation).

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