

Dynamic Analyses Using VAR Model with Mixed Frequency Data through Observable Representation*

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This article discusses dynamic analyses using the vector autoregressive (VAR) model for mixed-frequency data. The model estimation is achieved by representing the original model just with current and lagged observable variables. Such representation is accomplished through recursive substitution of unobservable variables with lagged observable variables. The consistent estimation of model parameters is facilitated by the classical minimum distance estimation that uses lagged variables as instruments. Conventional dynamic analyses, which include forecasting with the VAR model, are possible after model estimation. The proposed method differs from other approaches in three aspects. First, unlike a Bayesian approach, the proposed classical method does not require any specific prior distribution of coefficients. Second, an “explicit” identification condition is suggested for the model. Finally, the proposed method can estimate the error variance consistently, which is critical for dynamic analyses.

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I. Introduction

Standard vector autoregressive (VAR) models assume that the vectors of the data are observed at equispaced intervals of time (or the same least common multiples of frequency). However, economic variables are observed with different frequencies, which often restrict the utilization of VAR analysis.

For instance, with respect to the usual frequency of monetary policy decisions,

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central banks are interested in dynamic analyses including forecasting at the monthly level. However, the GDP and current account (target variables of monetary policy) are observed quarterly; thus, it is conventional to conduct impulse response analysis using a quarterly model. The result may therefore be inexact, while a monthly VAR analysis would seem appropriate for catching the dynamic effects of monetary policy shock. A most realistic approach would be to estimate a monthly VAR using these unconventional data. As another example, the market ending price in two different markets (i.e. daily stock price indices of the US and Japan) cannot be observed simultaneously due to time lag. Basically, important/discarded information may remain in the mixed-frequency data for more precise analyses.

The development of models for variables sampled at different frequencies has attracted substantial interest in the recent econometric literature. See Foroni and Giuseppe (2013) for an overview of the most common techniques. For instance, in forecasting, various works have exploited mixed-frequency data. Abeyasinghe (1998), Miller and Chin (1996), Nunes (2005), Ingentino-Trehan (1996), Webb (1999), Zheng and Rossiter (2006) are important examples. Ghysels, Santa-Clara and Valkanov (2004) introduced mixed data sampling (MIDAS) regression models, but they did not focus on the (Vector) autoregressive models.¹ MIDAS models specify conditional expectations as a distributed lag of regressors recorded at some higher sampling frequencies. Clements and Galvao (2006) found that use of monthly data through the MIDAS on the current quarter leads to significant improvement in forecasting current and next quarter output growth of the US. Kuzin, Marcellino and Schumacher (2009) compare the mixed-data sampling (MIDAS) and mixed-frequency VAR approaches to model specification in the presence of mixed-frequency data; e.g., monthly and quarterly series. In particular, they compare their performance in a relevant case for policy making; i.e., nowcasting and forecasting quarterly GDP growth in the euro area, on a monthly basis and using a set of 20 monthly indicators. It turns out that the two approaches are more complementary than substitutes, since mixed-frequency VAR tends to perform better for longer horizons, and MIDAS for shorter horizons.² Recently, Frank and Song (2013) developed a model that is cast in state-space form and estimated with Bayesian methods under a Minnesota-style prior. Eraker, et al. (2014) developed a Gibbs sampler for estimating VAR models with mixed and irregularly sampled data in a Bayesian context.

¹ "MIDAS involves regressors with different sampling frequencies, and are therefore not autoregressive models, since the notion of autoregression implicitly assumes that the data are sampled at the same frequency in the past." Ghysels et al. (2004, p. 1).

² Their approach is different from with ours because they follow Mariano and Murasawa (2003, 2007) and replace all missing values with zeros, where the missing values are assumed to be realizations of some i.i.d. standard normal random variable.

However, to date, most research seeking to exploit mixed frequency data is heavily focused on the forecasting and Bayesian approach. The Bayesian approach depends on a specific prior distribution of autoregressive coefficients (e.g., typically Minnesota prior) and aggregation observability of variables (e.g., a quarterly GDP is observable as an aggregation of unobservable monthly GDP). When a quarterly GDP is observable as an aggregation of unobservable monthly GDP, a maximum-likelihood estimation via the Kalman filtering method may be considered as a non-Bayesian approach. Zadzorny (1990) is an example in this direction. However, as correctly pointed out in Chen and Zadzorny (1998, pp. 47-48), it obscures the algebra of the model and the conditions under which it is identified and performs poorly or not at all on large models with many parameters.³

This limitation of maximum-likelihood estimation via Kalman filtering caused Chen and Zadzorny (1998) to propose a linear instrumental variable method as an alternative to compensate for missing data arising from the mixed frequencies through the use of Yule-Walker equations. Cadzow (1982) proposed and illustrated a closely related method. However, that method may not estimate a consistent error variance. Therefore, it cannot be used for typical dynamic analyses, including impulse response analysis and variance decomposition.

Due to the limitations of the aforementioned approaches, we suggest an estimation of the model by representation of the observable variables of the original model using the backward substitution of unobservable variables by the lagged observable variables. In this representation, the dependent and explanatory variables of the equation are all observable. The order of this representation may be exactly finite for a VAR (1) model, and approximately finite for a stationary VAR model, in the sense that largely lagged variables have near-zero coefficients. A standard test is also suggested for the order selection. The autoregressive coefficients and error variance matrix may be estimated by the *classical minimum distance* (CMD) using the lagged variables as instruments. Conventional dynamic analyses including forecasting with the VAR model are possible after model estimation.

Our approach differs from other approaches as follows. First, our classical approach does not require any specific prior distribution of coefficients as in a Bayesian approach. Second, we suggest an ‘explicit’ identification condition for the model (c.f., Theorem 4.1). Finally, our method can estimate the error variance consistently, which is critical for dynamic analyses.

The rest of the paper is organized as follows. Section II provides a discussion of the VAR model and data structure. In Section III, an instrumental variable

³ See a trivial example in Hamilton (pp. 387-388) that suggests a case where the parameters of the state-space representation are unidentified. In there, an observable variable y_t is an aggregation of two variables $y_t = \varepsilon_{1t} + \varepsilon_{2t}$ that represents an observation equation for a Kalman filtering where unobservable ε_{1t} and ε_{2t} are mean zero white noises with the variances σ_1^2 and σ_2^2 respectively. Under a Gaussian distribution assumption, neither σ_1^2 nor σ_2^2 is identified.

estimation of the observable variable representation is introduced. Section IV provides the CMD estimation of model parameters. In Section V, we discuss on the generalized impulse response analysis and model selection. An empirical illustration of the developed method is presented in Section VI. Finally, the paper is concluded in Section VII.

Throughout the paper, standard notations are used without any explicit references. In particular, \xrightarrow{p} and \xrightarrow{d} are used to signify the convergence in probability and in distribution, respectively, of random sequences.

II. VAR Model and Data Structure

Let z_t denote a $k \times 1$ vector of variables, assumed to be generated by a VAR(m) process,

$$z_t = \alpha_0 + (\alpha_1, \alpha_2, \dots, \alpha_m)Z_{t-1} + u_t; \quad (1)$$

or

$$Z_t = A_0 + AZ_{t-1} + U_t; \quad (2)$$

for $t = 0, 1, \dots, T$ where $Z_t = (z_t', z_{t-1}', \dots, z_{t-m+1}')'$, $U_t = (u_t', 0_{(m-1)k})'$,

$$A_0 = \begin{pmatrix} \alpha_0 \\ 0_{(m-1)k} \end{pmatrix} \text{ and } A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ I_k & 0_{kk} & \cdots & 0_{kk} \\ 0_{kk} & I_k & & \vdots \\ \vdots & & \ddots & \\ 0_{kk} & \vdots & & I_k & 0_{kk} \end{pmatrix},$$

where α_0 is a $k \times 1$ vector of constants, α_i ($i = 1, 2, \dots, m$) is a $k \times k$ matrix of coefficients, I_k denotes the k -dimensional identity matrix, 0_k is a $k \times 1$ zero vector, 0_{kk} is a $k \times k$ zero matrix, $Z_t = (z_t', z_{t-1}', \dots, z_{t-m+1}')'$ and u_t is assumed to be a $k \times 1$ vector with an unobserved mean of zero and independent and identically distributed disturbances with a positive definite covariance matrix, $\Sigma = (\sigma_{ij}) > 0$; $i, j = 1, 2, \dots, k$.

Further, the VAR model (1) is assumed to operate at a basic, fixed time interval normalized to unity. In each basic period, the process generates a value for each variable. Each variable is observed repeatedly at some regular time interval, which is an integral multiple of the basic interval. Thus i -th variable of z_t, z_{it} , $i = 1, 2, \dots, k$ may be observed at its own different frequency, ϕ_i . First, let us define frequency π as the least common multiple (LCM) of $\{\phi_1, \phi_2, \dots, \phi_k\}$. Further,

$n \equiv [T / \pi]$ is defined as the largest among all integers less than or equal to T / π , which is the sample number for estimation.

Our approach may be applied for following various cases of mixed frequency observations:

Example 2.1 Consider a VAR model of $z_t = (z_{1t}', z_{2t}')'$ with $k = k_1 + k_2$ are given as follows:

(a) *Monthly -quarterly observable data*

[Table 1] Monthly sample observation of variables

	0	1	2	3	4	5	6	7	8
z_{1t}	x	x	x	x	x	x	x	x	x
z_{2t}	x	o	o	x	o	o	x	o	o

Note: x denotes observable and o denotes unobservable variables.

For instance, z_{1t} can be quarterly growth of monthly observable financial variables (e.g., M1 and interest rate, $k_1 = 2$) while z_{2t} includes quarterly growth of a quarterly observable (at the end of a month) variable (e.g., GDP and current account, $k_2 = 2$), where $\pi = 3$.

(b) *Weekly-monthly observable data*

[Table 2] Weekly sample observation of variables

	0	1	2	3	4	5	6	7	8
z_{1t}	x	x	x	x	x	x	x	x	x
z_{2t}	x	o	o	o	x	o	o	o	o

For instance, z_{1t} includes weekly observable variables (e.g., federal funds rate and M1), z_{2t} can be a monthly observable real variables (e.g., industrial production index and unemployment rate) where $\pi = 4$.

(c) *Observable by turns data*

[Table 3] Daily sample observation of variables

	0	1	2	3	4	5	6	7	8
z_{1t}	x	o	x	o	x	o	x	o	x
z_{2t}	o	x	o	x	o	x	o	x	o

where the variable vector z_{1t} denotes a daily stock price index of the US and z_{2t} denotes a stock price index of Japan. The market prices of both markets cannot be observed simultaneously because of time lag.^{4,5} ■

For instance, with regard to the usual monetary policy decision frequency, central banks are interested in the dynamic analysis at the monthly level. However, since the GDP is observed quarterly, dynamic analysis using a quarterly model is conventional. As the results may be inexact, a monthly VAR analysis seems to be more appropriate. See the following example.

Example 2.2 (a) Consider a two-dimensional VAR model (1) of $z_t \equiv (z_{1t}, z_{2t})'$ with coefficient

$$\alpha \equiv \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$

where z_{1t} is observed monthly (e.g., money) and z_{2t} is observed quarterly (e.g., GDP). The LCM frequency, π , should therefore be one quarter in a monthly frequency model. Then, z_{1t} Granger causes z_{2t} if $\alpha_{12} \neq 0$. Assume that z_{t+3i} ; $i=1,2,\dots,n$ are all observable. The above model (1) is then transformed as a quarterly one:

$$z_t = \bar{\alpha} z_{t-3} + v_t \quad (3)$$

where $\bar{\alpha} \equiv \alpha^3$ and $v_t \equiv \alpha^2 u_{t-2} + \alpha u_{t-1} + u_t$. An OLS (ordinary least square) regression of coefficient $\bar{\alpha}$ is feasible, given as:

$$\hat{\bar{\alpha}} \equiv \left(\sum_{i=1}^n z_{t-3i} z_{t-3i}' \right)^{-1} \left(\sum_{i=1}^n z_{t-3i} z_t \right) \xrightarrow{p} \bar{\alpha}$$

which is consistent. If we assume $\alpha_{11}^2 + \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21} + \alpha_{22}^2 = 0$ even though $\alpha_{12} \neq 0$, then

$$\bar{\alpha} \equiv \begin{pmatrix} \alpha_{11}^3 + \alpha_{11}\alpha_{12}\alpha_{21} + \alpha_{12}\alpha_{21}(\alpha_{11} + \alpha_{22}) & 0 \\ 0 & \alpha_{22}^3 + \alpha_{12}\alpha_{21}\alpha_{22} + \alpha_{12}\alpha_{21}(\alpha_{11} + \alpha_{22}) \end{pmatrix}.$$

⁴ There is a 10-hour time lag between New York and Tokyo.

⁵ The impulse response to a shock that occurred during market closure may be computed from this VAR model.

So we may conclude z_{1t} does not Granger cause z_{2t} if we use the quarterly model (3), and the results may be different from those obtained using a monthly model. In short, if analysis is conducted solely based on the quarterly model (3), it may yield misleading results if a monthly model is more appropriate. ■

Chen and Zadrozny (1998) proposed a linear instrumental variable method that compensates for the missing data arising from mixed frequencies through use of Yule-Walker equations. However, their method may not estimate a consistent error variance, Σ , which means that it is not possible to carry out typical dynamic analyses such as impulse response analysis and variance decomposition. See the following example.

Example 2.3 For explanation, consider a VAR (1) model as $z_t = \alpha z_{t-1} + u_t$. Under the model stationarity, we may write:

$$Ez_t z_t' = \sum_{i=0}^{\infty} \alpha^i \Sigma \alpha^{i'} \text{ or } vec(\Sigma) = (\sum_{i=0}^{\infty} \alpha^i \otimes \alpha^{i'})^{-1} vec(Ez_t z_t'). \quad (4)$$

Chen and Zadrozny (1998) suggest an estimator of Σ as $vec(\bar{\Sigma}_K) = (\sum_{i=0}^K \hat{\alpha}^i \otimes \hat{\alpha}^{i'})^{-1} vec(E\hat{z}_t z_t')$ with a finite K , where $\hat{\alpha}$ and $E\hat{z}_t z_t'$ are estimators of α and $Ez_t z_t'$, respectively. Note that we cannot guarantee $\bar{\Sigma}_K \xrightarrow{p} \Sigma$ as $n \rightarrow \infty$. More seriously, if the model becomes non-stationary, then even representation (4) is not defined. ■

III. IV Estimation of Observable Variable Representation

In this section, an observable representation for the VAR model is derived, for which the dependent and explanatory variables are (asymptotically) observable autoregressive variables and error terms. Before proceeding, let us define $p_t' [a_t \times k]$ as a selection matrix⁶ which selects all observable variables from z_t at time t , and $P_{t-i}' [a_{t-i} \times mk]$ as a selection matrix which selects all observable variables from Z_{t-i} at time $t-i$. Similarly, let us define a selection matrix for the unobservable variables of Z_{t-i} as $Q_{t-i}' [b_{t-i} \times mk]$, where $mk = a_{t-i} + b_{t-i}$ for any i . For these matrices, the following results hold.

Proposition 3.1 (a) $P_{t-i} P_{t-i}' + Q_{t-i} Q_{t-i}' = I_{mk}$ for any i . (b) The selection matrices

⁶ In this paper, we define $A [b \times c]$ as a b by c matrix A . Note a_t and b_t denote the numbers of observable and unobservable variables of z_t respectively.

are invariant with frequency π as $P_{t-i} = P_{t-i \pm j\pi}$ and $Q_{t-i} = Q_{t-i \pm j\pi}$ for any $j = 1, 2, \dots, n$.

All proofs are in the Appendix.

Define the selected coefficients and variables as $\alpha_{0p,t} \equiv p_t' \alpha_0 [a_t \times 1]$, $\alpha_{pp,t-i} \equiv p_t' (\alpha_1, \alpha_2, \dots, \alpha_m) P_{t-i} [a_t \times a_{t-i}]$, $\alpha_{pq,t-i} \equiv p_t' (\alpha_1, \alpha_2, \dots, \alpha_m) Q_{t-i} [a_t \times b_{t-i}]$, $A_{0q,t-i} \equiv Q_{t-i}' A_0 [b_{t-i} \times mk]$, $A_{pp,t-i} \equiv P_{t-i+1}' A P_{t-i} [a_{t-i+1} \times a_{t-i}]$, $A_{pq,t-i} \equiv P_{t-i+1}' A Q_{t-i} [a_{t-i+1} \times b_{t-i}]$, $A_{qp,t-i} \equiv Q_{t-i+1}' A P_{t-i} [b_{t-i+1} \times a_{t-i}]$, $A_{qq,t-i} \equiv Q_{t-i+1}' A Q_{t-i} [b_{t-i+1} \times b_{t-i}]$, $p_t' z_t \equiv z_{p,t} [a_t \times 1]$, $P_{t-i}' Z_{t-i} \equiv Z_{p,t-i} [a_{t-i} \times 1]$, $Q_{t-i}' Z_{t-i} \equiv Z_{q,t-i} [b_{t-i} \times 1]$, $p_t' u_t \equiv u_{p,t} [a_t \times 1]$, $P_{t-i}' U_{t-i} \equiv U_{p,t-i} [a_{t-i} \times 1]$ and $Q_{t-i}' U_{t-i} \equiv U_{q,t-i} [b_{t-i} \times 1]$.

Then, the above VAR model (1) can be rewritten as:

$$\begin{aligned} z_{p,t} &= p_t' \alpha_0 + p_t' (\alpha_1, \alpha_2, \dots, \alpha_m) (P_{t-1} P_{t-1}' + Q_{t-1} Q_{t-1}') Z_{t-1} + p_t' u_t \\ &= \alpha_{0p,t} + \alpha_{pp,t-1} Z_{p,t-1} + \alpha_{pq,t-1} Z_{q,t-1} + u_{p,t} \end{aligned} \quad (5)$$

using $I_{mk} = P_{t-1} P_{t-1}' + Q_{t-1} Q_{t-1}'$ from Proposition 3.1. Equation (5) decomposes the explanatory variables into observable and unobservable parts. The lagged unobservable explanatory variable $Z_{q,t-1}$ in (5) may be written as:

$$\begin{aligned} Z_{q,t-i} &= Q_{t-i}' Z_{t-i} = Q_{t-i}' (A_0 + A Z_{t-i-1} + U_{t-i}) \\ &= Q_{t-i}' A_0 + Q_{t-i}' A (P_{t-i-1} P_{t-i-1}' + Q_{t-i-1} Q_{t-i-1}') Z_{t-i-1} + Q_{t-i}' U_{t-i} \\ &= A_{0q,t-i} + A_{qp,t-i-1} Z_{p,t-i-1} + A_{qq,t-i-1} Z_{q,t-i-1} + U_{q,t-i} \end{aligned} \quad (6)$$

for $i=1, 2, 3, \dots$ using $I_{mk} = P_{t-i-1} P_{t-i-1}' + Q_{t-i-1} Q_{t-i-1}'$. Then, equation (6) may be sequentially written for each $i=1, 2, 3, \dots, h-1$ as;

$$\begin{aligned} Z_{q,t-1} &= A_{0q,t-1} + A_{qp,t-2} Z_{p,t-2} + A_{qq,t-2} \underline{Z_{q,t-2}} + U_{q,t-1}, \\ \underline{Z_{q,t-2}} &= A_{0q,t-2} + A_{qp,t-3} Z_{p,t-3} + A_{qq,t-3} \underline{Z_{q,t-3}} + U_{q,t-2}, \\ \underline{Z_{q,t-3}} &= A_{0q,t-3} + A_{qp,t-4} Z_{p,t-4} + A_{qq,t-4} \underline{Z_{q,t-4}} + U_{q,t-3}, \\ &\vdots \\ \underline{Z_{q,t-h+1}} &= A_{0q,t-h+1} + A_{qp,t-h} Z_{p,t-h} + A_{qq,t-h} \underline{Z_{q,t-h}} + U_{q,t-h+1}. \end{aligned} \quad (7)$$

Now, a repetitive backward substitution of the unobservable variables for $\underline{Z_{q,t-i}}$ underlined in (7) results in the following equation;

$$\begin{aligned}
Z_{q,t-1} &= A_{0q,t-1} + A_{qq,t-2}A_{0q,t-2} + A_{qq,t-2}A_{qq,t-3}A_{0q,t-3} + \cdots + A_{qq,t-2}A_{qq,t-3} \cdots A_{qq,t-h+1}A_{0q,t-h+1} \\
&+ A_{qp,t-2}Z_{p,t-2} + A_{qq,t-2}A_{qp,t-3}Z_{p,t-3} + A_{qq,t-2}A_{qq,t-3}A_{qp,t-4}Z_{p,t-4} + \cdots \\
&+ A_{qq,t-2}A_{qq,t-3} \cdots A_{qq,t-h+1}A_{qp,t-h}Z_{p,t-h} \\
&+ A_{qq,t-2}A_{qq,t-3} \cdots A_{qq,t-h+1}A_{qq,t-h}Z_{q,t-h} \\
&+ A_{qq,t-2}A_{qq,t-3} \cdots A_{qq,t-h+1}U_{q,t-h+1} + \cdots + A_{qq,t-2}A_{qq,t-3}U_{q,t-3} + A_{qq,t-2}U_{q,t-2} + U_{q,t-1}.
\end{aligned} \tag{8}$$

If we plug the term $Z_{q,t-1}$ of (8) into (5), then the following form can be obtained:

$$z_{p,t} = \beta_{0,t} + \sum_{i=1}^h \beta_{pi,t} Z_{p,t-i} + \beta_{qh,t} Z_{q,t-h} + \varepsilon_{h,t} \tag{9}$$

where $\beta_{0,t} \equiv \alpha_{p,0} + \alpha_{pq,t-1}(A_{0q,t-1} + A_{qq,t-2}A_{0q,t-2} + A_{qq,t-2}A_{qq,t-3}A_{0q,t-3} + \cdots + A_{qq,t-2}A_{qq,t-3} \cdots A_{qq,t-h+1}A_{0q,t-h+1})[a_t \times 1]$, $\beta_{p1,t} \equiv \alpha_{pp,t-1}[a_t \times a_{t-1}]$, $\beta_{p2,t} \equiv \alpha_{pq,t-1}A_{qp,t-2}[a_t \times a_{t-2}]$, $\beta_{p3,t} \equiv \alpha_{pq,t-1}A_{qq,t-2}A_{qp,t-3}[a_t \times a_{t-3}]$, $\beta_{qi,t} \equiv \alpha_{pq,t-1}(\prod_{j=2}^i A_{qq,t-j})A_{qp,t-i}[a_t \times a_{t-i}]$ for $i \geq 4$, $\beta_{q1,t} \equiv \alpha_{pq,t-1}[a_t \times b_{t-1}]$, $\beta_{q2,t} \equiv \alpha_{pq,t-1}A_{qq,t-2}[a_t \times b_{t-2}]$, $\beta_{qi,t} \equiv \alpha_{pq,t-1}(\prod_{j=2}^{i-1} A_{qq,t-j})A_{qq,t-i}[a_t \times b_{t-i}]$ for $i \geq 3$ and $\varepsilon_{h,t} \equiv \sum_{i=1}^{h-1} \beta_{qi,t}U_{q,t-i} + u_{p,t}[a_t \times 1]$.

Note that $z_{p,t}$ may be represented by all observable explanatory variables in model (9) as $h \rightarrow \infty$.⁷ Further note that the coefficient of the unobservable variable in model (9), $\beta_{qh,t}$, approaches to zero as h becomes larger if the VAR model is stationary.⁸

Corollary 3.2 Suppose Model (1) is stationary, as coefficient A in model (1) is a real matrix ($mk \times mk$) with eigenvalues of $\lambda_1, \lambda_2, \dots, \lambda_{mk}$, which all have a modulus of less than 1. That is, $|\lambda_i| < 1$, for $i=1, \dots, r$. Then,⁹

$$z_{p,t} = \beta_{0,t} + \sum_{i=1}^h \beta_{pi,t} Z_{p,t-i} + \varepsilon_{h,t} + o(h^{-1}).$$

$\begin{matrix} a_t \times 1 & a_t \times 1 & a_t \times a_{t-i} & a_{t-i} \times a_t & a_t \times 1 \end{matrix}$

Stronger results hold than those of Corollary 3.2 for a VAR (1) model as:¹⁰

Lemma 3.3 Suppose all elements of $z_{p,t}$ are observable with frequency π and

⁷ A VAR(∞) model is discussed in Lewis and Reinsel (1985) where its estimation is conducted by OLS estimation. However the error term $\varepsilon_{h,t}$ in equation (9) is correlated with the explanatory variable $Z_{p,t-i}$ in our case and thus we later suggest an IV estimation of equation (9).

⁸ In this paper, we will assume a stationary VAR model as in Corollary 3.2.

⁹ We may omit h from $\varepsilon_{h,t}$ in following contents if it is necessary for any notational convenience.

¹⁰ Kim and Park (2007) consider a similar observable representation with a mixed frequency data. However their approach is just restricted to a simple VAR (1) model.

$m=1$. Then equation (9) is simplified as $z_t = \beta_{0,t} + \sum_{i=1}^{\pi} \beta_{pi,t} z_{p,t-i} + \varepsilon_{\pi,t}$.

See the following example.

Example 3.4 It is assumed that the observation structure in Table 1 of Example 2.1 and all k -elements of z_t are observable.

(a) Consider a VAR (1) model for which the coefficient matrix is given as:

$z_t = \alpha_0 + \alpha z_{t-1} + u_t$ where

$$\alpha_0 = \begin{pmatrix} \alpha_{01} \\ \alpha_{02} \end{pmatrix} \text{ and } \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}.$$

Therefore, there are k_1 observable and k_2 unobservable variables, respectively, where $k = k_1 + k_2$. Then, the representative sample selection matrices are given as $p_t' = I_k$, $P_{t-1}' = (I_{k_1} \ 0)$, $Q_{t-1}' = (0 \ I_{k_2})$, $P_{t-2}' = (I_{k_1} \ 0)$, $Q_{t-2}' = (0 \ I_{k_2})$, $P_{t-3}' = I_k$ and $Q_{t-3}' = 0$.

Using these selection matrices, we get

$$\begin{aligned} \alpha_{pp,t-1} &= I_k \alpha (I_{k_1} \ 0)' = \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix}, \quad \alpha_{pq,t-1} = I_k \alpha (0 \ I_{k_2})' = \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix}, \\ A_{qp,t-2} &= Q_{t-1}' \alpha P_{t-2} = (0 \ I_{k_2}) \alpha (I_{k_1} \ 0)' = \alpha_{21}, \quad A_{qq,t-2} = Q_{t-1}' \alpha Q_{t-2} = \alpha_{22}, \\ A_{qp,t-3} &= Q_{t-2}' \alpha P_{t-3} = (\alpha_{21} \ \alpha_{22}), \\ z_{p,t-1} &= P_{t-1}' z_{t-1} = (I_{k_1} \ 0) z_{t-1}, \quad z_{p,t-2} = P_{t-2}' z_{t-2} = (I_{k_1} \ 0) z_{t-2}, \\ z_{p,t-3} &= P_{t-3}' z_{t-3} = z_{t-3}, \\ \beta_{0,t} &= \alpha_0 + \alpha_{q,t-1} (Q_{t-1}' + A_{qq,t-2} Q_{t-2}') \alpha_0 = \alpha_0 + \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix} [(0 \ I_{k_2}) + \alpha_{22} (0 \ I_{k_2})] \alpha_0 \\ &= \begin{pmatrix} I_{k_1} & \alpha_{12} \alpha_{22} \\ 0 & I_{k_2} + \alpha_{22}^2 \end{pmatrix} \alpha_0 = \begin{bmatrix} \alpha_{01} + \alpha_{12} \alpha_{02} + \alpha_{12} \alpha_{22} \alpha_{02} \\ (I_{k_2} + \alpha_{22} + \alpha_{22}^2) \alpha_{02} \end{bmatrix}, \\ \beta_{p1,t} &= \alpha_{p,t-1} = \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix}, \quad \beta_{p2,t} = \alpha_{q,t-1} A_{qp,t-2} = \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix} \alpha_{21}, \\ \beta_{p3,t} &= \alpha_{q,t-1} A_{qq,t-2} A_{qp,t-3} = \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix} \alpha_{22} (\alpha_{21} \ \alpha_{22}), \quad \beta_{q1,t} = \alpha_{q,t-1} = \alpha Q_{t-1} = \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix} \end{aligned}$$

$$\text{and } \beta_{q2,t} = \alpha_{q1,t} A_{qq,t-2} = \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix} \alpha_{22}.$$

The final observable representation is given as:

$$z_t = \beta_0 + \beta_{p1,t-1} z_{p,t-1} + \beta_{p2,t-2} z_{p,t-2} + \beta_{p3,t-3} z_{p,t-3} + \varepsilon_t \quad (10)$$

where $\varepsilon_t \equiv \beta_{q2,t} u_{q,t-2} + \beta_{q1,t} u_{q,t-1} + u_t$.

(b) For a VAR (2) model, the sample selection matrices for $t = 3i, i = 0, 1, 2, \dots$ are given as:¹¹

$$\begin{aligned} P_{t-1}' &= \begin{pmatrix} I_{k_1} & 0 & 0 & 0 \\ 0 & 0 & I_{k_1} & 0 \end{pmatrix} \text{ and } Q_{t-1}' = \begin{pmatrix} 0 & I_{k_2} & 0 & 0 \\ 0 & 0 & 0 & I_{k_2} \end{pmatrix}; \\ P_{t-2}' &= \begin{pmatrix} I_{k_1} & 0 & 0 & 0 \\ 0 & 0 & I_{k_1} & 0 \\ 0 & 0 & 0 & I_{k_1} \end{pmatrix} \text{ and } Q_{t-2}' = \begin{pmatrix} 0 & I_{k_2} & 0 & 0 \end{pmatrix}; \\ P_{t-3}' &= \begin{pmatrix} I_{k_1} & 0 & 0 & 0 \\ 0 & I_{k_1} & 0 & 0 \\ 0 & 0 & I_{k_1} & 0 \end{pmatrix} \text{ and } Q_{t-3}' = \begin{pmatrix} 0 & 0 & 0 & I_{k_2} \end{pmatrix}. \end{aligned}$$

(c) For a VAR (3) model, the sample selection matrices for $t = 3i, i = 0, 1, 2, \dots$ are given as:

$$\begin{aligned} P_{t-1}' &= \begin{pmatrix} I_{k_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{k_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{k_1} \end{pmatrix} \text{ and } Q_{t-1}' = \begin{pmatrix} 0 & I_{k_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{k_2} & 0 & 0 \end{pmatrix}; \\ P_{t-2}' &= \begin{pmatrix} I_{k_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{k_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{k_1} & 0 \end{pmatrix} \text{ and } Q_{t-2}' = \begin{pmatrix} 0 & I_{k_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{k_2} \end{pmatrix}. \end{aligned}$$

¹¹ We do not need to compute the specific form of coefficient $\beta_{pi,t}$ in a computer coding for the estimation. The above sample selection matrices are sufficient for that purpose.

$$P_{t-3}' = \begin{pmatrix} I_{k_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{k_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{k_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{k_1} & 0 \end{pmatrix} \text{ and } Q_{t-3}' = \begin{pmatrix} 0 & 0 & 0 & I_{k_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{k_2} \end{pmatrix}. \blacksquare$$

For theoretical/economic reasons, zero elements of the VAR coefficient may often induce a finite order of observable representation. See the following example.

Example 3.5 Consider a VAR (1) model. Assuming the observation structure in Table 3 of Example 2.1, k_1 observable and k_2 unobservable variables appear, respectively, where $k = k_1 + k_2$. Then, representative sample selection matrices are given as $p_t' = (I_{k_1} \ 0)$, $p_{t-1}' = (0 \ I_{k_2})$, $p_{t-1-2i}' = (0 \ I_{k_2})$, $p_{t-2i}' = (I_{k_1} \ 0)$, $Q_{t-1-2i}' = (I_{k_1} \ 0)$, or $Q_{t-2i}' = (0 \ I_{k_2})$, for $i \geq 1$, respectively.

Using these selection matrices, we get

$$\begin{aligned} \alpha_{pp,t-1} &= (I_{k_1} \ 0) \alpha \begin{pmatrix} 0 \\ I_{k_2} \end{pmatrix} = \alpha_{12}, \quad \alpha_{pp,t-2} = (0 \ I_{k_2}) \alpha \begin{pmatrix} I_{k_1} \\ 0 \end{pmatrix} = \alpha_{21}, \\ \alpha_{pq,t-1} &= (I_{k_1} \ 0) \alpha \begin{pmatrix} I_{k_1} \\ 0 \end{pmatrix} = \alpha_{11}, \quad \alpha_{pq,t-2} = (0 \ I_{k_2}) \alpha \begin{pmatrix} 0 \\ I_{k_2} \end{pmatrix} = \alpha_{22}, \\ A_{qp,t-2} &= (I_{k_1} \ 0) \alpha \begin{pmatrix} I_{k_1} \\ 0 \end{pmatrix} = \alpha_{11}, \quad A_{qq,t-2} = (I_{k_1} \ 0) \alpha \begin{pmatrix} 0 \\ I_{k_2} \end{pmatrix} = \alpha_{12}, \\ A_{qp,t-3} &= (0 \ I_{k_2}) \alpha \begin{pmatrix} 0 \\ I_{k_2} \end{pmatrix} = \alpha_{22}, \quad A_{qq,t-3} = (0 \ I_{k_2}) \alpha \begin{pmatrix} I_{k_1} \\ 0 \end{pmatrix} = \alpha_{21}, \\ A_{qq,t-4} &= Q_{t-3}' \alpha Q_{t-4} = (I_{k_1} \ 0) \alpha \begin{pmatrix} 0 \\ I_{k_2} \end{pmatrix} = \alpha_{12}, \quad A_{qp,t-4} = (I_{k_1} \ 0) \alpha \begin{pmatrix} I_{k_1} \\ 0 \end{pmatrix} = \alpha_{11}, \\ A_{qp,t-5} &= (0 \ I_{k_2}) \alpha \begin{pmatrix} 0 \\ I_{k_2} \end{pmatrix} = \alpha_{22}, \quad \alpha_{0p,t} \equiv p_t' \alpha_0, \quad A_{0q,t-i} \equiv Q_{t-i}' \alpha_0, \\ \beta_{p1,t} &= \alpha_{pp,t-1} = \alpha_{12}, \quad \beta_{p2,t} = \alpha_{pq,t-1} A_{qp,t-2} = \alpha_{11} \alpha_{11}, \\ \beta_{p3,t} &= \alpha_{pq,t-1} A_{qq,t-2} A_{qp,t-3} = \alpha_{11} \alpha_{12} \alpha_{22}, \\ \beta_{p4,t} &= \alpha_{pq,t-1} A_{qq,t-2} A_{qq,t-3} A_{qp,t-4} = \alpha_{11} \alpha_{12} \alpha_{21} \alpha_{11}, \\ \beta_{q4,t} &= \alpha_{pq,t-1} A_{qq,t-2} A_{qq,t-3} A_{qq,t-4} = \alpha_{11} \alpha_{12} \alpha_{21} \alpha_{12}, \\ \beta_{p5,t} &= \alpha_{pq,t-1} A_{qq,t-2} A_{qq,t-3} A_{qq,t-4} A_{qp,t-5} = \alpha_{11} \alpha_{12} \alpha_{21} \alpha_{12} \alpha_{22}, \\ \beta_{q1,t} &= \alpha_{pq,t-1} = p_t' \alpha Q_{t-1} = \alpha_{11} \text{ and} \\ \beta_{q2,t} &= \alpha_{pq,t-1} A_{qq,t-2} = \alpha_{11} \alpha_{12}. \end{aligned}$$

If the stock price of Japan does not Granger cause that of the US (or $\alpha_{21} = 0$), then one observable representation is given as:

$$z_{p,t} = \beta_{0,t} + \beta_{p1,t} z_{p,t-1} + \beta_{p2,t} z_{p,t-2} + \beta_{p3,t} z_{p,t-3} + \varepsilon_t,$$

where $\beta_{0,t} \equiv \alpha_{p,0} + \alpha_{pq,t-1}(A_{0q,t-1} + A_{qq,t-2}A_{0q,t-2}) = \alpha_{01} + \alpha_{11}(\alpha_{01} + \alpha_{12}\alpha_{02})$ and $\varepsilon_t \equiv \beta_{q2,t}u_{q,t-2} + \beta_{q1,t}u_{q,t-1} + u_t$ because $\beta_{pi,t} = 0$ for $i > 3$ and $\beta_{q4,t} = 0$. Further the other observable representation is given as:

$$z_{p,t-1} = \tilde{\beta}_{0,t-1} + \tilde{\beta}_{p1,t-1} z_{p,t-2} + \tilde{\beta}_{p2,t-1} z_{p,t-3} + \varepsilon_t,$$

where $\tilde{\beta}_{p1,t-1} = \alpha_{pp,t-2} = \alpha_{21}$, $\tilde{\beta}_{p2,t-1} = \alpha_{pq,t-2}A_{qp,t-3} = \alpha_{22}^2$, $\tilde{\beta}_{p3,t-1} = \alpha_{pq,t-2}A_{qq,t-3}$, $A_{qp,t-4} = \alpha_{22}\alpha_{21}\alpha_{11}$, $\tilde{\beta}_{q1,t} = \alpha_{pq,t-2} = p_{t-1}'\alpha Q_{t-2} = \alpha_{22}$, $\tilde{\beta}_{q3,t-1} = \alpha_{pq,t-2}A_{qq,t-3}A_{qq,t-4} = \alpha_{22}\alpha_{21}\alpha_{12}$, $\tilde{\beta}_{0,t-1} = \alpha_{p,t-1} + \alpha_{pq,t-2}A_{0q,t-2} = \alpha_{02} + \alpha_{22}\alpha_{02}$ because $\beta_{pi,t} = 0$ for $i > 2$ and $\beta_{q3,t-1} = 0$ with $\varepsilon_t \equiv \tilde{\beta}_{q1,t}u_{q,t-1} + u_t$. ■

Now, let us assume that the dimension of the explanatory variable is fixed as h . Then, separating the time of explanatory variables periodically, model (9) can be rewritten as:

$$z_{p,t} = \gamma_t' W_t + \varepsilon_t \text{ for } t = \pi, 2\pi, \dots, n\pi, \quad (11)$$

where $W_t = (1, z_{p,t-1}', z_{p,t-2}', \dots, z_{p,t-h-m+1}')' [(1 + \sum_{i=1}^{h+m-1} a_{t-i}) \times 1]$, $\gamma_t' [(1 + \sum_{i=1}^{h+m-1} a_{t-i}) \times a_t]$ is a coefficient conformably defined for W_t . Note that a vector of variables $X_t = (1, z_{p,t-h}', z_{p,t-h-1}', \dots, z_{p,t-h-d}')' [(1 + \sum_{i=0}^d a_{t-h-i}) \times 1]$ is not correlated with ε_t , and is thus a candidate of the instrument where W_t and ε_t are correlated with each other; and d is selected to satisfy an inequality $\sum_{i=1}^{h+m-1} a_{t-i} \leq \sum_{i=0}^d a_{t-h-i}$ as an order condition for identification.¹²

Then, for estimation, we can exploit that the coefficients and components of the dependent and explanatory variables in model (11) are time-invariant with frequency π ; i.e., the time-dependent coefficient may be written as a constant:

$$\gamma_\pi = \dots = \gamma_{n\pi} \equiv \gamma. \quad (12)$$

¹² To select a number of instruments optimally with a finite/small sample has been discussed in literatures. For instance, Donald and Newey (2001) suggest an approximate MSE to select the number of instrument for a given set of instruments of which approximated form is derived using higher order asymptotic theory. However we now just focus on the large sample result or the consistency and thus the small sample properties of our estimator are remained as a further research topic.

Then, a stacked form of (11) can be given as:

$$z_p = W\gamma + \varepsilon [n \times a_t], \quad (13)$$

where $z_p = (z_{p,\pi}, z_{p,2\pi}, \dots, z_{p,n\pi})'$, $W = (W_\pi, W_{2\pi}, \dots, W_{n\pi})'$, $\varepsilon = (\varepsilon_\pi, \varepsilon_{2\pi}, \dots, \varepsilon_{n\pi})'$ and $X = (X_\pi, X_{2\pi}, \dots, X_{n\pi})'$. For estimation of model (13), let us define an instrumental variable (IV) estimation of γ as:

$$\hat{\gamma} = (W'P_X W)^{-1} W'P_X z_p \quad [(1 + \sum_{i=1}^{h+m-1} a_{t-i}) \times a_t], \quad (14)$$

assuming $|W'P_X W| \neq 0$ almost surely where $P_X = X(X'X)^{-1}X'$.

Then, the asymptotic property of the IV estimator in (14) can be derived as:

Theorem 3.6 Suppose Model (1) is stationary and (u_t) is an i.i.d. process with a finite fourth moment. Then

$$n^{1/2}[\text{vec}(\hat{\gamma} - \gamma)] \xrightarrow{d} N(0, HDH')$$

$$\text{as } n \rightarrow \infty \text{ where } H = p \lim \left[I_{a_t} \otimes \left(\frac{W'X}{n} \left(\frac{X'X}{n} \right)^{-1} \frac{X'W}{n} \right)^{-1} \frac{W'X}{n} \left(\frac{X'X}{n} \right)^{-1} \right] \text{ and}$$

$$D = p \lim \frac{\varepsilon' \varepsilon}{n} \otimes p \lim \frac{X'X}{n}.$$

A Wald test on the null hypothesis $H_0 : \beta_{qh+1,t} = 0$ is also possible indirectly. This null implies that model (9) does not have unobservable variables with a finite order. So the IV estimator of coefficients in (14) may be consistent. To test this null, we further assume

Assumption 3.7 $\beta_{ph+1,t} = 0$ implies that $\beta_{qh+1,t} = 0$.

Assumption 3.8 may be justified because the two terms share the common term $\alpha_{q,t-1} \left(\prod_{j=2}^{h+1} A_{qq,t-j} \right) P_{t-h+1} A$, which approaches to zero as h becomes larger (or exactly zero, c.f., Example 3.5) under the situation of Corollary 3.2. Under Assumption 3.7, it is sufficient to test another null $H_0 : \beta_{ph+1,t} = 0$ to check the original null.

For construction of the test statistic on the alternative null, note that the coefficient $\beta_{ph+1,t}$ is matched with the variables $Z_{p,t-h-1} = (z_{p,t-h-1}, z_{p,t-h-2}, \dots, z_{p,t-h-m})'$. Then, the following test statistic can be defined:

$$\tau_n \equiv n[Cvec(\hat{\gamma})]' \text{var}(CH'DHC)^{-1} Cvec(\hat{\gamma})$$

where $Cvec(\gamma) = vec(\beta_{ph+1,t})$ with

$$C = \begin{pmatrix} 0 & \sum_{i=h+1}^{h+m} a_{t-i} \times (1 + \sum_{i=1}^h a_{t-i}) & I_{\sum_{i=h+1}^{h+m} a_{t-i}} & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \sum_{i=h+1}^{h+m} a_{t-i} \times (1 + \sum_{i=1}^h a_{t-i}) & I_{\sum_{i=h+1}^{h+m} a_{t-i}} & 0 & \vdots \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & \cdots & & \cdots & 0 & I_{\sum_{i=h+1}^{h+m} a_{t-i}} \end{pmatrix}$$

and its limit distribution as;

Theorem 3.8 Suppose Assumption 3.7 holds. Then, under the null $H_0 : \beta_{ph+1,t} = 0$, we get

$$\tau_n \xrightarrow{d} \chi^2_{a_t \sum_{i=h+1}^{h+m} a_{t-i}}.$$

IV. CMD Estimation of Model Parameters

Now the autoregressive coefficient as $\alpha \equiv vec(\alpha_0, \alpha_1, \dots, \alpha_m)$ is estimated using a CMD estimation. A CMD estimator of $\hat{\alpha}$ is defined by a solution to the following problem:

$$\min_{\alpha} \pi(\alpha) \tag{15}$$

where $vec(\alpha) \in R^{mk^2+k}$ and $\pi(\alpha) \equiv vec[\hat{\gamma} - \gamma(\alpha)]' vec[\hat{\gamma} - \gamma(\alpha)]$. So, the difference between $\hat{\gamma}$ and $\gamma(\alpha)$ is made to be as small as possible.

The consistency of the above CMD estimator $\hat{\alpha}$ that is a solution of problem (15) is given as:

Theorem 4.1 Suppose $\nabla vec[\gamma(\alpha^*)]$ has a full column rank, where $\alpha^* \in (\hat{\alpha}, \tilde{\alpha})$ and $\nabla vec[\gamma(\alpha^*)] \equiv \frac{\partial vec[\gamma(\alpha)]}{\partial \alpha'} \Big|_{\alpha=\alpha^*}$ is the gradient of γ at α^* . Then $\hat{\alpha} \xrightarrow{p} \tilde{\alpha}$, where $\tilde{\alpha}$ is a true value of α .

Note that (i) a semi-inequality $(1 + \sum_{i=1}^{h+m-1} a_{t-i})a_t \geq mk^2 + k$ which represents that the number of elements in γ should be larger than or equal to the number of

autoregressive coefficients in equation (1) and (ii) $vec[\gamma(\alpha)]$ should include all elements of α are necessary conditions for $\nabla vec[\gamma(\alpha^*)]$ to have a full column rank in Theorem 4.1.

Then, the above CMD estimator has following limit distribution:

Theorem 4.2 Suppose $\hat{\alpha} \xrightarrow{p} \tilde{\alpha}$. Assume $G'G$ is nonsingular where $G \equiv \nabla vec[\gamma(\tilde{\alpha})]$. Then $n^{1/2}[vec(\hat{\alpha}) - vec(\tilde{\alpha})] \xrightarrow{d} N[0, (G'G)^{-1}G'HDH'G(G'G)^{-1}]$.

Now, suppose all elements of α are identified. Then the variance of the error term Σ in equation (1) can be estimated assuming $a_t = k$ or the dependent variables in (11) are all observable. To accomplish this, from equation (9), we derive

$$\begin{aligned} \Omega \equiv E\varepsilon_t \varepsilon_t' &= \sum_{i=1}^{h-1} \beta_{qi,t} Q_{t-i}' \begin{pmatrix} \Sigma & 0_{k \times (m-1)k} \\ 0_{(m-1)k \times k} & 0_{(m-1)k \times (m-1)k} \end{pmatrix} Q_{t-i} \beta_{qi,t}' + p_t' \Sigma p_t \quad \text{for } t = i\pi, i = 1, 2, 3, \dots \\ &= \sum_{i=0}^{h-1} R_i' \Sigma R_i \end{aligned} \quad (16)$$

where $R_0 = p_t = I_k$ and $R_i' \equiv \beta_{qi,t} Q_{t-i}' \times \begin{pmatrix} I_k \\ 0_{(m-1)k \times k} \end{pmatrix}$ for $i = 1, 2, \dots, h-1$.

Now, from the vectorization of (16), we get

$$vec(\Omega) = \left[\sum_{i=0}^{h-1} (R_i' \otimes R_i') \right] vec(\Sigma) \quad (17)$$

or

$$vec(\Sigma) = \left[\sum_{i=0}^{h-1} (R_i' \otimes R_i') \right]^{-1} vec(\Omega) \quad (18)$$

assuming $\sum_{i=0}^{h-1} (R_i' \otimes R_i')$ is not singular.

Note that R_i' is a function of the VAR coefficient α , and thus a plug-in estimator $R_i'(\hat{\alpha})$ is conformably defined. Now, we define the estimator of Σ as:

$$vec(\hat{\Sigma}) = \left[\sum_{i=0}^{h-1} (R_i'(\hat{\alpha}) \otimes R_i(\hat{\alpha})) \right]^{-1} vec(\hat{\Omega}) \quad (19)$$

from (18), where $R_i(\hat{\alpha})' \equiv \beta_{qi,t}(\hat{\alpha}) Q_{t-i}' \times \begin{pmatrix} I_k \\ 0_{(m-1)k \times k} \end{pmatrix}$ for $i = 1, 2, \dots, h-1$, $\hat{\varepsilon} = z - W\hat{\gamma}$ and $\hat{\Omega} = \frac{\hat{\varepsilon}' \hat{\varepsilon}}{n}$.

This estimator is consistent as:

Corollary 4.3 Suppose $\hat{\alpha} \xrightarrow{p} \tilde{\alpha}$ and $\sum_{i=0}^{h-1} (R_i' \otimes R_i')$ is not singular. Then $vec(\hat{\Sigma}) \xrightarrow{p} vec(\Sigma)$.

Example 4.4 Assume that VAR (1) model in Example 3.4 holds. An IV estimation is next defined as

$$\hat{\gamma} = (W' P_X W)^{-1} W' P_X z$$

where $W_t = (1, z_{p,t-1}', z_{p,t-2}', z_{t-3}')' [(1+2k_l + k) \times 1]$ and $X_t = (1, z_{p,t-3}', z_{p,t-4}', \dots, z_{p,t-10}')' [(1+3k+5k_l) \times 1]$. Note that a necessary condition for rank in Theorem 4.1 is satisfied as $k(2k_l + k) \geq k(k+1)$ for all $k_l \geq 0$.

Define a Wald test for the null hypothesis $H_0 : \beta_{p,t-4} = 0$

$$\tau_n = n[Cvec(\hat{\gamma})]' var\left(\hat{C}\hat{H}'\hat{D}\hat{H}\hat{C}'\right)^{-1} Cvec(\hat{\gamma}) \xrightarrow{d} \chi_{kk_l}^2$$

which checks whether VAR order is one or not, where $m=1$, $h=3$, $a_{t-4} = k_l$, $\sum_{i=1}^3 a_{t-i} = 2k_l + k$ and

$$C = \begin{pmatrix} 0 & I_{k_l} & 0 & \dots & 0 & 0 \\ k_l \times (1+2k_l+k) & & & & & \\ \vdots & & 0 & I_{k_l} & & 0 \\ & k_l \times (1+2k_l+k) & & & & \\ 0 & & & & & \vdots \\ 0 & 0 & \dots & \dots & 0 & 0 & I_{k_l} \\ & & & & k_l \times (1+2k_l+k) & & \end{pmatrix},$$

$$\hat{H} = nI_k \otimes (W' P_X W)^{-1} W' X (X' X)^{-1}, \quad \hat{\varepsilon} = z - W\hat{\gamma} \quad \text{and} \quad \hat{D} = \frac{\hat{\varepsilon}' \hat{\varepsilon}}{n} \otimes \frac{X' X}{n}.$$

$$\text{Note } vec[\gamma(\alpha)] = \begin{pmatrix} \alpha_{01} + \alpha_{12} \alpha_{22} \alpha_{02} \\ (I_{k_2} + \alpha_{22}^2) \alpha_{02} \\ vec(\alpha_{11}) \\ vec(\alpha_{21}) \\ vec(\alpha_{12} \alpha_{21}) \\ vec(\alpha_{22} \alpha_{21}) \\ vec(\alpha_{12} \alpha_{22} \alpha_{21}) \\ vec(\alpha_{22}^2 \alpha_{21}) \\ vec(\alpha_{12} \alpha_{22}^2) \\ vec(\alpha_{22}^3) \end{pmatrix}$$

from equation (10) and its Jacobian matrix is derived as

$$G \equiv \frac{\partial \nabla \gamma(\alpha)}{\partial [\alpha_{01}', \alpha_{02}', \text{vec}(\alpha_{11})', \text{vec}(\alpha_{21})', \text{vec}(\alpha_{12})', \text{vec}(\alpha_{22})']}$$

$$= \begin{pmatrix} I_{k_1} & \alpha_{12} \alpha_{22}' & 0 & 0 & \alpha_{02}' \alpha_{22}' \otimes I_{k_1} & \alpha_{02}' \otimes \alpha_{12} \\ k_1 \times k_1 & k_1 \times k_2 & k_1 \times k_1^2 & k_1 \times k_1 k_2 & k_1 \times k_1 k_2 & k_1 \times k_2^2 \\ 0 & I_{k_2} + \alpha_{22}^2 & 0 & 0 & 0 & (\alpha_{02}' \otimes I_{k_2})(\alpha_{22}' \otimes I_{k_2} + I_{k_2} \otimes \alpha_{22}) \\ k_2 \times k_1 & k_2 \times k_2 & k_2 \times k_1^2 & k_2 \times k_1 k_2 & k_2 \times k_1 k_2 & k_2 \times k_2^2 \\ 0 & 0 & I_{k_1}^2 & 0 & 0 & 0 \\ k_1^2 \times k_1 & k_1^2 \times k_2 & k_1^2 \times k_1^2 & k_1^2 \times k_1 k_2 & k_1^2 \times k_1 k_2 & k_1^2 \times k_2^2 \\ 0 & 0 & 0 & I_{k_1} k_2 & 0 & 0 \\ k_1 k_2 \times k_1 & k_1 k_2 \times k_2 & k_1 k_2 \times k_1^2 & k_1 k_2 \times k_1 k_2 & k_1 k_2 \times k_1 k_2 & k_1 k_2 \times k_2^2 \\ 0 & 0 & 0 & I_{k_1} \otimes \alpha_{12} & \alpha_{21}' \otimes I_{k_1} & 0 \\ k_1^2 \times k_1 & k_1^2 \times k_2 & k_1^2 \times k_1^2 & k_1^2 \times k_1 k_2 & k_1^2 \times k_1 k_2 & k_1^2 \times k_2^2 \\ 0 & 0 & 0 & I_{k_1} \otimes \alpha_{22} & 0 & \alpha_{21}' \otimes I_{k_2} \\ k_1 k_2 \times k_1 & k_1 k_2 \times k_2 & k_1 k_2 \times k_1^2 & k_1 k_2 \times k_1 k_2 & k_1 k_2 \times k_1 k_2 & k_1 k_2 \times k_2^2 \\ 0 & 0 & 0 & I_{k_1} \otimes \alpha_{12} \alpha_{22} & \alpha_{21}' \alpha_{22}' \otimes I_{k_1} & \alpha_{21}' \otimes \alpha_{12} \\ k_1^2 \times k_1 & k_1^2 \times k_2 & k_1^2 \times k_1^2 & k_1^2 \times k_1 k_2 & k_1^2 \times k_1 k_2 & k_1^2 \times k_2^2 \\ 0 & 0 & 0 & I_{k_1} \otimes \alpha_{22}^2 & 0 & (\alpha_{21}' \otimes I_{k_2})(\alpha_{22}' \otimes I_{k_2} + I_{k_2} \otimes \alpha_{22}) \\ k_1 k_2 \times k_1 & k_1 k_2 \times k_2 & k_1 k_2 \times k_1^2 & k_1 k_2 \times k_1 k_2 & k_1 k_2 \times k_1 k_2 & k_1 k_2 \times k_2^2 \\ 0 & 0 & 0 & 0 & \alpha_{22}^2 \otimes I_{k_1} & (I_{k_2} \otimes \alpha_{12})(\alpha_{22}' \otimes I_{k_2} + I_{k_2} \otimes \alpha_{22}) \\ k_1 k_2 \times k_1 & k_1 k_2 \times k_2 & k_1 k_2 \times k_1^2 & k_1 k_2 \times k_1 k_2 & k_1 k_2 \times k_1 k_2 & k_1 k_2 \times k_2^2 \\ 0 & 0 & 0 & 0 & 0 & (\alpha_{22}')^2 \otimes I_{k_2} + \alpha_{22}' \otimes \alpha_{22} + I_{k_2} \otimes \alpha_{22}^2 \\ k_2^2 \times k_1 & k_2^2 \times k_2 & k_2^2 \times k_1^2 & k_2^2 \times k_1 k_2 & k_2^2 \times k_1 k_2 & k_2^2 \times k_2^2 \end{pmatrix}$$

using the following properties (Lütkepohl, 1993);

(i) if δ is an $m \times 1$ vector, $A(\delta)$ is $n \times p$ and $B(\delta)$ is $p \times q$, then

$$\frac{\partial \text{vec}(AB)}{\partial \delta'} = (I_q \otimes A) \frac{\partial \text{vec}(B)}{\partial \delta'} + (B' \otimes I_n) \frac{\partial \text{vec}(A)}{\partial \delta'},$$

$$(ii) \quad \frac{\partial \text{vec}(A^r)}{\partial \delta'} = \left[\sum_{i=0}^{r-1} (A')^{r-1-i} \otimes A^i \right] \frac{\partial \text{vec}(A)}{\partial \delta'},$$

$$(iii) \quad \text{vec}(ABC) = (C' \otimes A) \text{vec}(B) \quad \text{and} \quad \text{vec}(AB) = (B' \otimes I) \text{vec}(A) = (I \otimes A) \text{vec}(B).$$

Then, usual inferences on the coefficients $[\alpha_{01}', \alpha_{02}', \text{vec}(\alpha_{11})', \text{vec}(\alpha_{21})', \text{vec}(\alpha_{12})', \text{vec}(\alpha_{22})']$ including a Granger causality test may be conducted. Finally, the estimator of error variance is obtained as:

$$\text{vec}(\hat{\Sigma}) = \left[\sum_{i=1}^2 (\hat{R}_i' \otimes \hat{R}_i) + I_{k^2} \right]^{-1} \text{vec}(\hat{\Omega}) \quad (20)$$

$$\text{where } \hat{R}_1' \equiv \begin{pmatrix} \hat{\alpha}_{12} \\ \hat{\alpha}_{22} \end{pmatrix} Q_{t-1}', \quad \hat{R}_2' \equiv \begin{pmatrix} \hat{\alpha}_{12} \\ \hat{\alpha}_{22} \end{pmatrix} \hat{\alpha}_{22} Q_{t-2}', \quad \text{and} \quad \hat{\Omega} \equiv \frac{1}{n} \hat{\varepsilon}' \hat{\varepsilon}. \quad \blacksquare$$

Remark 4.5 (a) Note the CMD estimator of autoregressive coefficient α is

determined by a solution of problem (15) and thus, if any, its possible bias may closely depend on the bias of $\hat{\gamma}$. We may readily show that the asymptotic bias of IV estimator $\hat{\gamma}$ due to $\beta_{qh,t} \neq 0$ (i.e., the coefficient for unobservable variable in equation (9)) is given as:

$$\begin{aligned} p \lim (\hat{\gamma} - \gamma) &= (p \lim \frac{W' P_X W}{n})^{-1} p \lim \frac{W' X}{n} (p \lim \frac{X' X}{n})^{-1} p \lim \frac{X' Z_{q-h}}{n} \beta_{qh,t}, \\ &= o(h^{-1}) \end{aligned} \quad (21)$$

where the second equality comes from Corollary 3.2 and all other terms are $O_p(1)$. Therefore, note a proportional increasing of h with the sample number n deserves the consistency of $\hat{\gamma}$ as:

$$\hat{\gamma} - \gamma = o_p(1) \quad (22)$$

if $h = n^\delta$ where $\delta > 0$ is a constant. At least we may reduce the possible bias of $\hat{\gamma}$ by increasing h while its marginal effect (and the sensitivity of estimated parameters) is decreasing. Increased VAR order m requires conformably increased h to reduce possible bias of model estimators. For instance, note $h = \pi$ for VAR (1) model while $h > \pi$ for VAR ($m > 1$) model (c.f., Lemma 3.3). The increase of ‘non-stationarity’ of model may reduce the rate of decrease of $\beta_{qh,t}$ and thus requires conformably increased h (c.f., Corollary 3.2).

(b) To select the dimension k , order m and lag length h of VAR model, conventional Akaike and Schwarz information criterion of forecast precision may be considered. If we assume that unobservable variables do not exist in (1) and Gaussian error distribution, Schwarz criterion (SC) is strongly consistent while Akaike criterion (AC) is not. See Lütkepohl (1993, pp. 128-138) for the details including small sample properties of these. These criteria depend on a penalty through the addition of unnecessary parameters (in its second term) while those may not decrease $\ln(|\Sigma|)$ (in its first term). Note our estimator $\hat{\Sigma}$ is consistent as a maximum likelihood (ML) estimator even if it is not an ML estimator itself. Therefore we expect that AC and SC may work in our modeling structure.

However, those may not be directly applied to our case because our estimator is not a maximum-likelihood estimator with a Gaussian error distribution. To select the order m and lag length h of VAR model, we recommend use of the τ_n test. For instance, a larger $m > 1$ implies a large tail in a observable form when it is compared with VAR (1) model (c.f., Lemma 3.3). ■

V. Generalized Impulse Response Analysis and Model Selection

After the estimation of VAR model, we may conduct the generalized impulse analysis from Pesaran and Shin (1998), which yields unique impulse response functions that are invariant to the ordering of variables:

$$\psi_j^g(i) = \sigma_{jj}^{-1/2} \kappa^i \Sigma e_j, i = 1, 2, \dots, n \quad (23)$$

which measures the effect of one standard error shock to the j -th equation at time t on the expected values of z at time $t+n$, where e_j is a $k \times 1$ selection vector with unity as its j -th element and zeros elsewhere, where

$$\kappa_i = \alpha_1 \kappa_{i-1} + \alpha_2 \kappa_{i-2} + \dots + \alpha_m \kappa_{i-m} \quad (24)$$

with $\kappa_0 = I_k$ and $\kappa_i = 0$ for $i < 0$.

Note we may write the recursive form solution of (24) as:

$$\kappa_i = \bar{e}' A^i \bar{e}, \quad (25)$$

because

$$\kappa_i = \bar{e}' \begin{pmatrix} \kappa_i \\ \kappa_{i-1} \\ \vdots \\ \kappa_{i-m+1} \end{pmatrix} = \bar{e}' \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ I_k & 0_{kk} & \cdots & 0_{kk} \\ 0_{kk} & I_k & & \vdots \\ \vdots & & \ddots & \\ 0_{kk} & \vdots & & I_k & 0_{kk} \end{pmatrix} \begin{pmatrix} \kappa_{i-1} \\ \kappa_{i-2} \\ \vdots \\ \kappa_{i-m} \end{pmatrix} \text{ and } \begin{pmatrix} \kappa_0 \\ \kappa_{-1} \\ \vdots \\ \kappa_{-m+1} \end{pmatrix} = \bar{e}$$

from (24) for the second equality and definition where $\bar{e} = I_k$ for $m=1$ and

$$\bar{e} = \begin{pmatrix} I_k \\ 0_{(m-1)k} \end{pmatrix} \text{ for } m \geq 2.$$

Now we derive the asymptotic distribution of the generalized impulse functions:

Theorem 5.1 Suppose that (u_t) is an i.i.d. process and model (1) is stationary where $E|u_t^4| < \infty$. Then $\hat{\psi}_j^g(i) - \psi_j^g(i) \xrightarrow{d} N(0, \Psi_j(i) \Gamma \Psi_j(i)')$ where

$$\Gamma \equiv E \left[\begin{pmatrix} \text{vec}(X_t \varepsilon_t') \\ \text{vec}[(\varepsilon_t \varepsilon_t') - \Omega] \end{pmatrix} \begin{pmatrix} \text{vec}(X_t \varepsilon_t') \\ \text{vec}[(\varepsilon_t \varepsilon_t') - \Omega] \end{pmatrix}' \right], \quad \alpha_v \equiv \text{vec}(\alpha_1,$$

$$\dots, \alpha_m) \quad , \quad G_v \equiv \nabla \text{vec}[\gamma(\alpha_v^*)] \equiv \frac{\partial \text{vec}[\gamma(\alpha)]}{\partial \alpha_v^*} \bigg|_{\alpha_v = \alpha_v^*} \quad , \quad \Lambda \equiv (\lambda_{1,1}, \lambda_{2,1}, \dots, \lambda_{k,1}, \lambda_{1,2}, \lambda_{2,2}, \dots, \lambda_{k,2}, \dots, \lambda_{1,mk^2}, \lambda_{2,mk^2}, \dots, \lambda_{k,mk^2}) \quad \text{with}$$

$$\lambda_{i,j} \equiv \frac{\partial \gamma_{i,j}}{\partial \alpha_{ij}} \quad$$

$$- [\sum_{\ell=0}^{h-1} (R_\ell'(\alpha) \otimes R_\ell(\alpha))]^{-1} \left[\sum_{\ell=0}^{h-1} \left(\frac{\partial R_\ell'(\alpha)}{\partial \alpha_{ij}} \otimes R_\ell'(\alpha) + R_\ell'(\alpha) \otimes \frac{\partial R_\ell'(\alpha)}{\partial \alpha_{ij}} \right) \right] [\sum_{\ell=0}^{h-1} (R_\ell'(\alpha) \otimes R_\ell(\alpha))]^{-1} \text{vec}(\Omega)$$

and

$$\Psi_j(i) \equiv - \frac{[(e_j' \Sigma \otimes I_k), (e_j' \Sigma \otimes \kappa_i) - \frac{1}{2} \sigma_{jj}^{-1} \psi_j^g(i) (e_j' \otimes e_j')]}{\sqrt{\sigma_{jj}}} \times$$

$$\begin{pmatrix} -(\bar{e}' \otimes \bar{e}') [\sum_{\ell=0}^{i-1} (A')^{i-1-\ell} \otimes A^\ell] \frac{\partial \text{vec}(A)}{\partial \alpha_v'} (G_v' G_v)^{-1} G_v' H & 0 \\ (\Lambda (G_v' G_v)^{-1} G_v' + [\sum_{\ell=0}^{h-1} (R_\ell' \otimes R_\ell)]^{-1} [(I_k \otimes p \lim \varepsilon' W / n) + (p \lim \varepsilon' W / n \otimes I_k) K]) H & [\sum_{\ell=0}^{h-1} (R_\ell' \otimes R_\ell)]^{-1} \end{pmatrix}$$

where K is a communication matrix such that $\text{vec}[(\gamma - \hat{\gamma})'] = K \text{vec}[(\gamma - \hat{\gamma})]$.

Remark 5.2 Note the variance term in the asymptotic distribution of the generalized impulse functions in Theorem 5.1 is complicated. Thus, with a small number of samples, we may use a bootstrap approach to compute it to construct a confidence interval. ■

VI. An Application for Global Impact of Monetary Policy Shock

In this section, dynamic analyses are conducted using a semi-global VAR model of the interest rate and GDP for the US, Japan and Euro area (17 countries). The monetary policies determining the interest rates in these countries are usually conducted monthly (or nearly so). Note that GDP is observed quarterly, while consumer price index (CPI) and industrial production index are observed monthly.

Here, we aim to analyze the dynamic effects of monetary policy with a monthly frequency. This would enable the policy makers in central banks to undertake more sophisticated reactions to the turbulence of financial economic markets than would be possible using quarterly models. This can provide an advantage if we consider the wild variation of the financial economy in recent days. Further, the global structure enables analysis of the effects of recent global expansionary monetary policies on the international economy on a monthly basis.

All variables were real-adjusted by CPI and log-differenced, except the interest rate. Data were obtained from Fred 2 of FRB, Saint Louis, and from the Statistical Data Warehouse of the European Central Bank. The GDP was seasonally adjusted; the interest rate indicated ‘interest rates for government securities and government

Bonds'; CPI was the Consumer Price Index for all urban consumers for the US, the harmonized index of consumer prices for the 17 Euro-area countries, and consumer price index of all items for Japan. The interest rate and CPI were not seasonally adjusted, while the industrial production index was seasonally adjusted. The sample periods for all available data were from January 1996 to January 2014 by quarter, and from January 1996 to March 2014 by month. This left a total of 73 samples, with 42 parameters.¹³

A VAR (1) model was considered as a base case¹⁴, for which the observable representation is given as:

$$z_t = \beta_0 + \beta_{1,t-1} z_{p,t-1} + \beta_{2,t-2} z_{p,t-2} + \beta_{3,t-3} z_{t-3} + \varepsilon_t$$

where z_t and z_{t-3} are composed of the real GDP growth rate and changes in the real interest rate, while $z_{p,t-1}$ and $z_{p,t-2}$ are just composed of changes in the real interest rates, respectively,

$$\begin{aligned} \beta_{0,t} &= \begin{bmatrix} \alpha_{01} + \alpha_{12} \alpha_{22} \alpha_{02} \\ (I_3 + \alpha_{22}^2) \alpha_{02} \end{bmatrix}, \quad \beta_{p1,t} = \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \end{pmatrix}, \quad \beta_{p2,t} = \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix} \alpha_{21}, \\ \beta_{p3,t} &= \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix} \alpha_{22} (\alpha_{21} \quad \alpha_{22}), \quad \beta_{q1,t} = \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix} \text{ and } \beta_{q2,t} = \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix} \alpha_{22} \text{ where} \\ (\alpha_{01}', \alpha_{02}') &= (a_{01}, a_{02}, a_{03}, a_{04}, a_{05}, a_{06})', \quad \alpha_{11} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \\ \alpha_{12} &= \begin{pmatrix} a_{14} & a_{15} & a_{16} \\ a_{24} & a_{25} & a_{26} \\ a_{34} & a_{35} & a_{36} \end{pmatrix}, \quad \alpha_{21} = \begin{pmatrix} a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{pmatrix} \text{ and } \alpha_{22} = \begin{pmatrix} a_{44} & a_{45} & a_{46} \\ a_{54} & a_{55} & a_{56} \\ a_{64} & a_{65} & a_{66} \end{pmatrix}. \end{aligned}$$

Then, an IV estimation can be defined as:

$$\hat{\gamma} = (W' P_X W)^{-1} W' P_X z$$

where $W_t = (1, z_{p,t-1}', z_{p,t-2}', z_{t-3}')' [13 \times 1]$ and $X_t = (1, z_{t-3}', z_{p,t-4}', z_{p,t-5}', z_{t-6}')' [16 \times 1]$. Further, the estimator of error variance may be computed as:

¹³ The transformation matrix from 219 monthly data points to 73 quarterly data points was $I_{73} \otimes \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ when the third element was selected from among three months in a quarter. The Gauss codes and data are available at 'http://blog.naver.com/yunyeongkim/Gausscode'.

¹⁴ It is considering a curse of dimension; i.e., we will have 78 parameters to be estimated in the case of a VAR (2) model with just 73 sample numbers. Of course, this primary specification will be tested later.

$$vec(\hat{\Sigma}) = [\sum_{i=1}^2 (\hat{R}_i' \otimes \hat{R}_i) + I_9]^{-1} vec(\hat{\Omega})$$

$$\text{where } \hat{R}_1' \equiv \begin{pmatrix} \hat{\alpha}_{12} \\ \hat{\alpha}_{22} \end{pmatrix} \begin{pmatrix} 0 & I_3 \end{pmatrix}_{3 \times 6}, \quad \hat{R}_2' \equiv \begin{pmatrix} \hat{\alpha}_{12} \\ \hat{\alpha}_{22} \end{pmatrix} \hat{\alpha}_{22} \begin{pmatrix} 0 & I_3 \end{pmatrix}_{3 \times 6} \quad \text{and} \quad \hat{\Omega} \equiv \frac{1}{71} \hat{\varepsilon}' \hat{\varepsilon}.$$

Then, a Jacobian matrix may be computed as:

$$G \equiv \frac{\partial \nabla \gamma(\alpha)}{\partial [\alpha_{01}', \alpha_{02}', vec(\alpha_{11})', vec(\alpha_{21})', vec(\alpha_{12})', vec(\alpha_{22})']_{78 \times 42}}$$

$$= \begin{pmatrix} I_3 & \alpha_{12} \alpha_{22} & 0 & 0 & \alpha_{02}' \alpha_{22}' \otimes I_3 & \alpha_{02}' \otimes \alpha_{12} \\ 0 & I_{k_2} + \alpha_{22}^2 & 0 & 0 & 0 & (\alpha_{02}' \otimes I_{k_2}) (\alpha_{22}' \otimes I_{k_2} + I_{k_2} \otimes \alpha_{22}) \\ 0 & 0 & I_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_9 & 0 & 0 \\ 0 & 0 & 0 & I_3 \otimes \alpha_{12} & \alpha_{21}' \otimes I_3 & 0 \\ 0 & 0 & 0 & I_3 \otimes \alpha_{22} & 0 & \alpha_{21}' \otimes I_{k_2} \\ 0 & 0 & 0 & I_{k_1} \otimes \alpha_{12} \alpha_{22} & \alpha_{21}' \alpha_{22}' \otimes I_{k_1} & \alpha_{21}' \otimes \alpha_{12} \\ 0 & 0 & 0 & I_{k_1} \otimes \alpha_{22}^2 & 0 & (\alpha_{21}' \otimes I_{k_2}) (\alpha_{22}' \otimes I_{k_2} + I_{k_2} \otimes \alpha_{22}) \\ 0 & 0 & 0 & 0 & \alpha_{22}^2 \otimes I_{k_1} & (I_{k_2} \otimes \alpha_{12}) (\alpha_{22}' \otimes I_{k_2} + I_{k_2} \otimes \alpha_{22}) \\ 0 & 0 & 0 & 0 & 0 & (\alpha_{22}')^2 \otimes I_{k_2} + \alpha_{22}' \otimes \alpha_{22} + I_{k_2} \otimes \alpha_{22}^2 \end{pmatrix}_{9 \times 9}$$

where

$$[\alpha_{01}', \alpha_{02}', vec(\alpha_{11})', vec(\alpha_{21})', vec(\alpha_{12})', vec(\alpha_{22})']_{42 \times 1} = [a_{01}, a_{02}, a_{03}, a_{04}, a_{05}, a_{06}, a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}, a_{13}, a_{23}, a_{33}, a_{41}, a_{51}, a_{61}, a_{42}, a_{52}, a_{62}, a_{43}, a_{53}, a_{63}, a_{14}, a_{24}, a_{34}, a_{15}, a_{25}, a_{35}, a_{16}, a_{26}, a_{36}, a_{44}, a_{54}, a_{64}, a_{45}, a_{55}, a_{65}, a_{46}, a_{56}, a_{66}]'$$

According to the τ_n test on the null hypothesis that the interest rate does not Granger cause the GDP, the null hypothesis cannot be rejected at the 5% level (*critical value* = 3.84). See Table 4 for the results.

[Table 4] Granger causality test results

	US GDP	EU GDP	Japan GDP
US interest rate \Rightarrow	$H_0 : a_{41} = 0$, <u>0.029</u>	$H_0 : a_{51} = 0$, <u>0.000</u>	$H_0 : a_{61} = 0$, <u>0.061</u>
EU interest rate \Rightarrow	$H_0 : a_{42} = 0$ <u>0.006</u>	$H_0 : a_{52} = 0$, <u>0.015</u>	$H_0 : a_{62} = 0$, <u>0.125</u>
Japan interest rate \Rightarrow	$H_0 : a_{43} = 0$, <u>0.001</u>	$H_0 : a_{53} = 0$, <u>0.429</u>	$H_0 : a_{63} = 0$, <u>0.023</u>

Note: The underlined value is a computed test statistic of the null hypothesis coefficient.

A test was also conducted to check the null hypothesis of VAR (1) vs. alternative VAR (2), which gave rise to the statistic 0.345. Therefore, the null hypothesis again cannot be rejected at the 5% significance level (*critical value* = 3.84).

Finally, the monthly VAR model was estimated, and the generalized impulse analysis was conducted

$$\psi_j^g(n) = \sigma_{jj}^{-1/2} \alpha^n \Sigma e_j ,$$

where $j=1,2,...,6$ and $n=1,2,...,10$ in a VAR (1) model which measures the effect of one standard error shock to the j -th equation at time t on the expected values of z at time $t+n$.

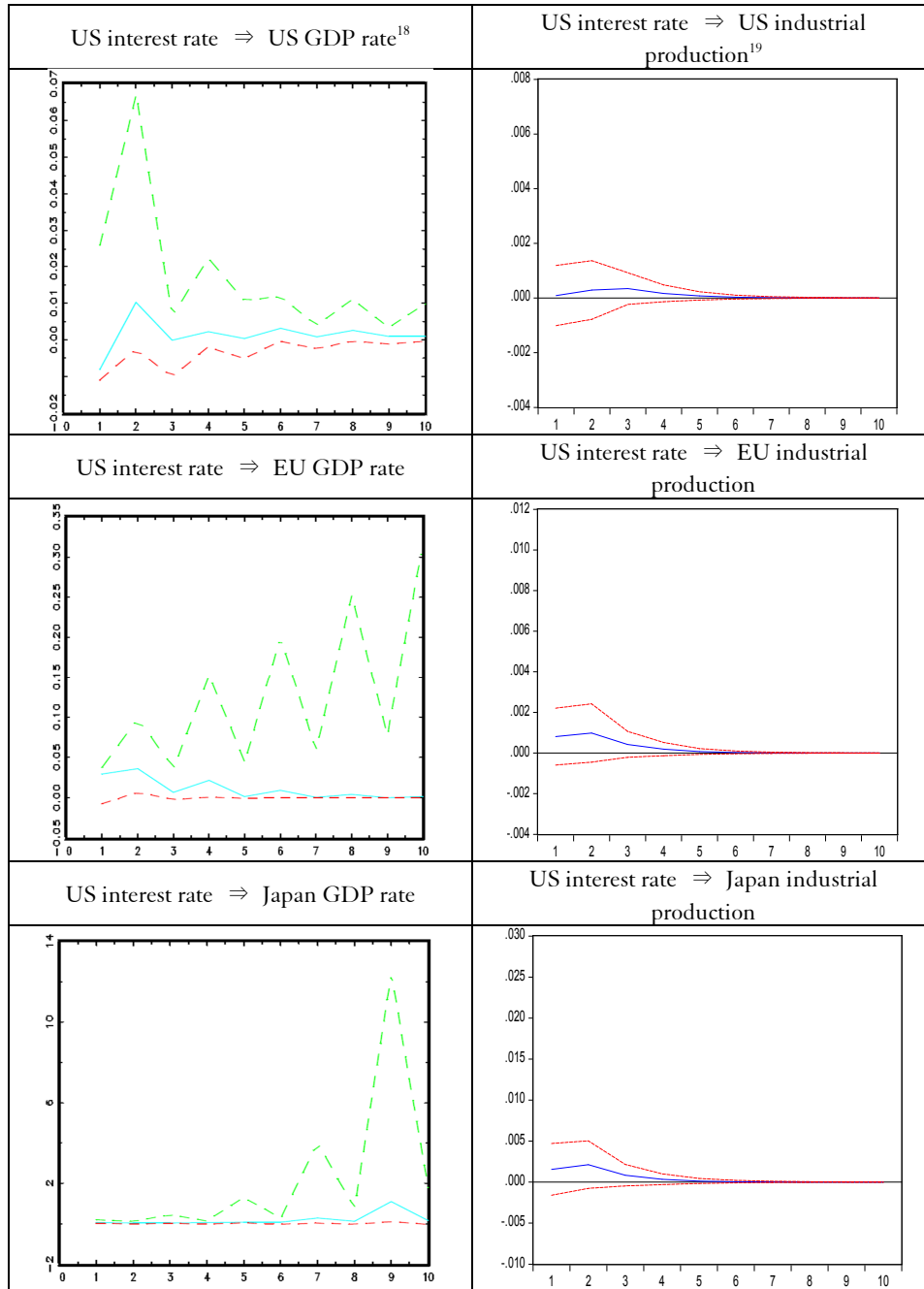
The impulse response analyses revealed several facts. First, the shock of increasing the real interest rate change in the US positively affected the real GDP growth rate of the US (computed by the log difference) for roughly three months. This was followed by a slight negative effect.¹⁵ Similar patterns could be observed for the EU, but not for Japan; i.e., while the shock of increasing real interest rate change in the EU positively affected the EU real GDP growth rate, as in the US, this was not true in Japan.¹⁶ The shock of increasing the real interest rate change in the EU negatively affected the rate of change of the real GDP in the US.¹⁷

To check the robustness of these results, a similar impulse response analysis was also conducted in a semi-global VAR model for the US, EU and Japan, composed of the same real interest rates and monthly real industrial production index change rates. Note that the industrial production index is often used as a substitute for the GDP (which is only observable quarterly), while the former usually does not cover

¹⁵ A possible explanation for the positive response of the GDP to increase in the real interest rate may be that the increase is accompanied with foreign capital inflow and induced augmentation of economic activity.

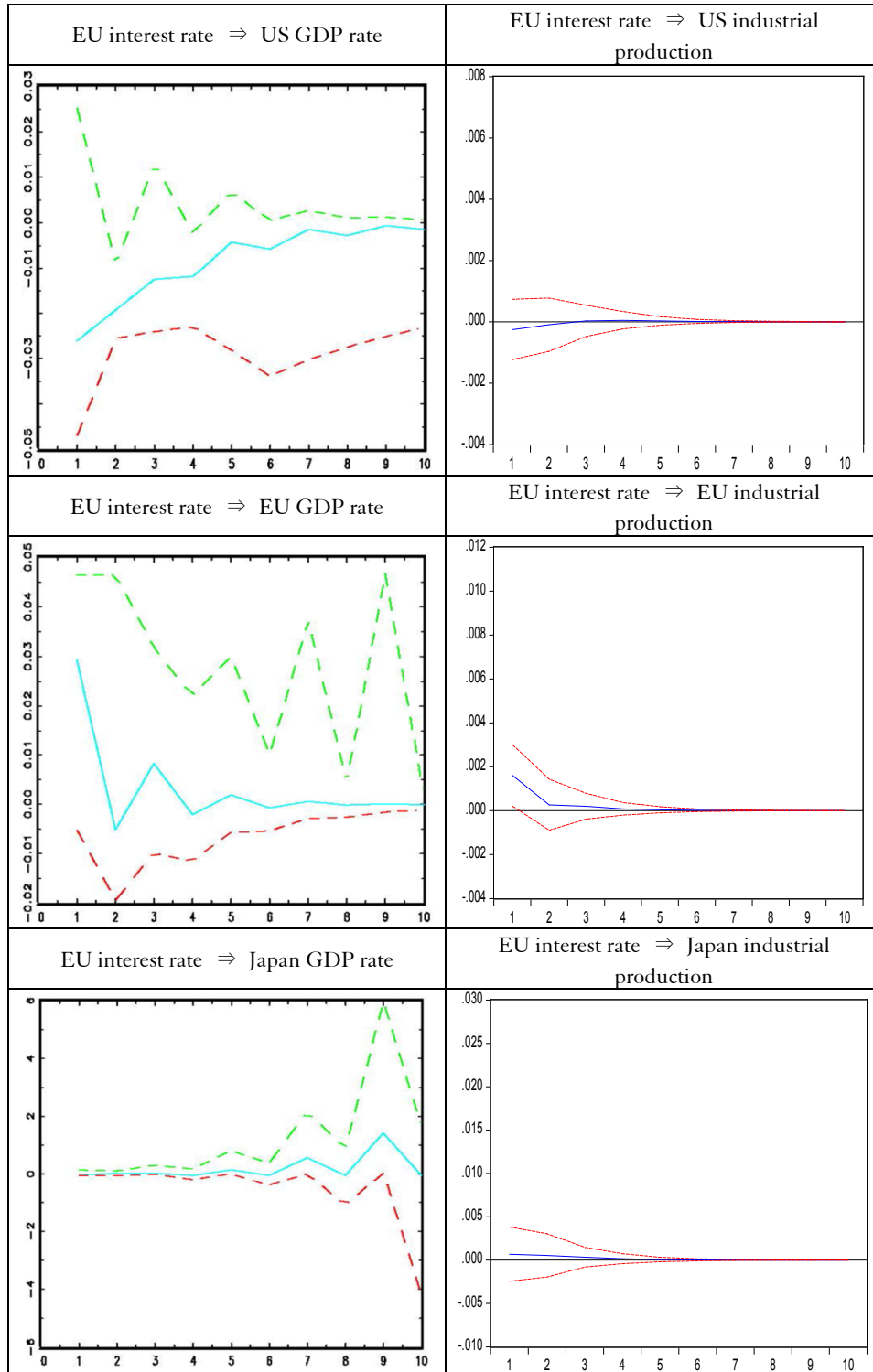
¹⁶ It seems that increase of the Japanese interest rate does not attract foreign investors as in the case of the US, which is expected considering the relatively small size of the economy.

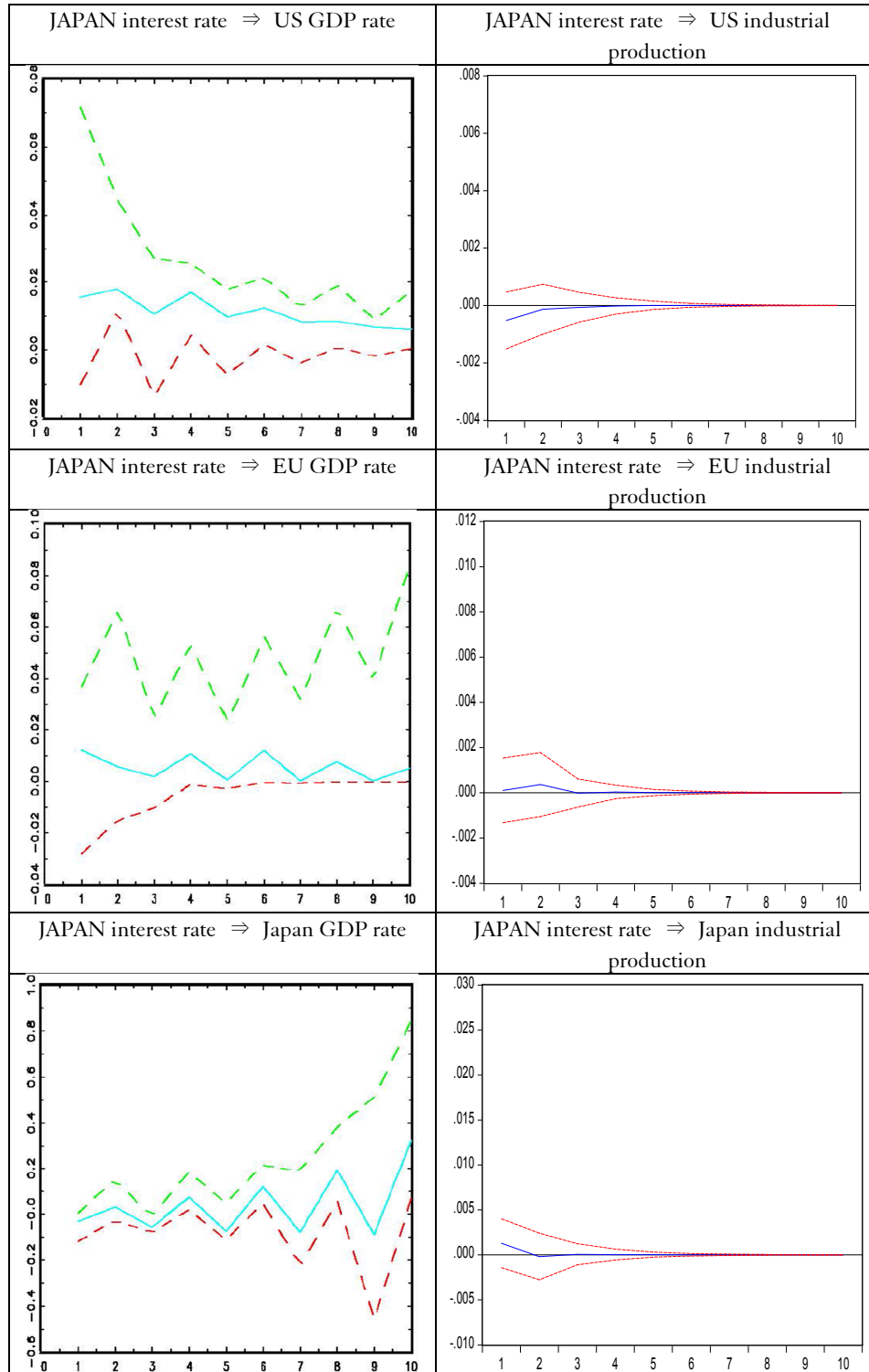
¹⁷ We suspect that this is also related to competition between the US and EU in the foreign capital market.

[Figure 1] Comparisons of Generalized Impulse Response: GDP and Industrial Production

¹⁸ The dotted lines are 40th and 60th quartiles and solid line are median of impulse responses from 300 bootstrap resamplings computed by GAUSS 7.0.

¹⁹ The dotted lines are 95% confidence intervals computed by Eviews 7.1.





the service sector.

In this experiment, similar patterns in the impulse response locus for the US and the EU area were observed, with differences only in the Japanese interest rate shock. See Figure 1 for the results. For instance, the shock of increasing the rate of real interest change in the US positively affected the rate of change in the US real industrial production index. Further, the shock of increasing the real interest rate change in the EU also negatively affected the change rate of the US real industrial production index. These results were also observed when the mixed frequency VAR model was employed.

VII. Conclusion

In this paper, dynamic analyses were generalized using a VAR model for mixed frequency data. Estimation of the model was possible through a representation of the observable variables of the original model using the backward substitution of unobservable variables by the lagged observable variables. In this representation, the dependent and explanatory variables of the equation were all observable. The order of the representation may be exactly finite for a VAR (1) model, and approximately finite for a stationary VAR model, in the sense that largely lagged variables have near-zero coefficients. A standard test was suggested for the order selection. The autoregressive coefficients and error variance matrix may be estimated by the CMD using the lagged variables as instruments. It was possible to carry out conventional dynamic analyses with the VAR model after model estimation.

Several important topics remain subjects for future research. First, the developed method may be applied to a non-stationary VAR model including an error correction model under standard regularity conditions. However, the distributions of estimators and test statistics are non-standard, as that requires simulation works. Second, small sample performances depending on the model parameter selection need to be evaluated. Finally, comparing forecasting performance with other alternatives including Bayesian and MIDAS is necessary.

Appendix A: Proofs of Theorems

Proposition 3.1: (a) To show this, suppose $P_t \equiv (e_1, e_2, \dots, e_a)$, where e_i denotes an mk -dimensional unit vector of which the i -th element is one and all other elements are zeros. Note that $P_t P_t'$ is a matrix in which the diagonal elements from the first to the a -th elements are ones while all other elements are zeros. Similarly, $Q_t Q_t'$ is a matrix in which the diagonal elements from the $a+1$ -th and to the mk -th elements are ones while all other elements are zeros, where $Q_t \equiv (e_{a+1}, \dots, e_{mk})$. Thus, the result $P_t P_t' + Q_t Q_t' = I_{mk}$ can be obtained. Even if the unit vectors of P_t are arranged in a different order, the same result holds.

(b) Note the observability structures are repeated with frequency π under assumption. Therefore, the claimed result holds. ■

Corollary 3.2: Note that we may write (9) as

$$\begin{aligned} z_{p,t} &= \beta_{0,t} + \sum_{i=1}^h \beta_{pi,t} Z_{p,t-i} + \beta_{qh,t} Z_{q,t-h} + \varepsilon_{h,t} \\ &= \beta_{0,t} + \sum_{i=1}^h \beta_{pi,t} Z_{p,t-i} + \sum_{j=h+1}^{\infty} \beta_{pj,t} Z_{p,t-j} + \sum_{i=1}^{h-1} \beta_{qi,t} U_{q,t-i} + \sum_{j=h}^{\infty} \beta_{qj,t} U_{q,t-j} + u_{p,t} \end{aligned}$$

after repetitive substitution. Further note

$$\sum_{j=h+1}^{\infty} \beta_{pj,t} Z_{p,t-j} < M \sum_{j=h+1}^{\infty} |\beta_{pj,t}|$$

for an $M(>0)$ because $Z_{p,t-j}$ is an $O_p(1)$ variable. However, note that $\sum_{j=h+1}^{\infty} |\beta_{pj,t}|$ is $o(h^{-1})$ from following facts (a) and (b):

$$\begin{aligned} h \sum_{j=h+1}^{\infty} |\beta_{pj,t}| &= h \sum_{j=h+1}^{\infty} |\alpha_{pq,t-1} (A_{qq,t-2} \cdots A_{qq,t-j+1}) A_{qp,t-j}| \\ &\leq h |\alpha_{pq,t-1}| \sum_{j=h+1}^{\infty} |(Q_{t-1}' A Q_{t-2} Q_{t-2}' A Q_{t-3} Q_{t-3}' A \cdots A Q_{t-j+2} Q_{t-j+2}' A Q_{t-j+1})| |A_{qp,t-j}| \\ \text{(a)} \quad &\leq h |\alpha_{pq,t-1} Q_{t-1}'| \sum_{j=h+1}^{\infty} |A^{j-1}| |A_{qp,t-j}| \\ &= |\alpha_{pq,t-1} Q_{t-1}'| |h A^h| \sum_{j=1}^{\infty} |A^j| |A_{qp,t-j}|, \end{aligned}$$

where the second inequality holds because the zero elements of $Q_{t-i} Q_{t-i}'$ in $A Q_{t-2} Q_{t-2}' A \cdots A Q_{t-j+2} Q_{t-j+2}' A$ just drive the elements of A^{j-1} closer to zero, where $Q_{t-i} Q_{t-i}'$ is an identity matrix ($mk \times mk$), except for some zero diagonal elements.

$$\text{(b)} \quad h \sum_{j=h+1}^{\infty} |\beta_{pj,t}| \longrightarrow 0$$

because (i) $\sum_{j=1}^{\infty} |A^j| < \infty$ from the stationarity assumption, and

$$(ii) \quad hA^h = hJ \begin{bmatrix} \Lambda_1^h & & 0 \\ & \Lambda_2^h & \\ & & \ddots \\ 0 & & & \Lambda_r^h \end{bmatrix} J^{-1} \rightarrow 0$$

for a conformable matrix J from the Jordan canonical form for equality and (a), where

$$\Lambda_i^h = \begin{bmatrix} \lambda_i^h & \binom{h}{1} \lambda_i^h & \cdots & \binom{h}{r_i-1} \lambda_i^h \\ & \lambda_i^h & & \binom{h}{r_i-2} \lambda_i^h \\ & & \ddots & \\ 0 & & & \lambda_i^h \end{bmatrix} \text{ for } i=1,2,\dots,r.$$

Similarly note that

$$\sum_{j=h}^{\infty} \beta_{qj,t} U_{q,t-i} = o_p(h^{-1})$$

and thus

$$z_{p,t} = \beta_{0,t} + \sum_{i=1}^h \beta_{pi,t} Z_{p,t-i} + \varepsilon_{h,t} + o_p(h^{-1})$$

where $\varepsilon_{ht} \equiv \sum_{i=1}^{h-1} \beta_{qi,t} U_{q,t-i} + u_{p,t}$. ■

Lemma 3.3: Note that $Z_t = z_t$ under VAR (1) in equation (1), and z_t is observable with a frequency of π from the assumption. Thus, equation (9) is revised as $z_t = \beta_{0,t} + \sum_{i=1}^{\pi} \beta_{pi,t} z_{p,t-i} + \beta_{q\pi,t} z_{q,t-\pi} + \varepsilon_{\pi,t}$ after replacing h by π . However note $\beta_{q\pi,t} = 0$ from $A_{qq,t-\pi} = 0$ using $Q_{t-\pi} = 0$ where $P_{t-\pi} = I_k$. Therefore the claimed result holds. ■

Theorem 3.7: Note from the definition

$$n^{1/2} \text{vec}(\hat{\gamma} - \gamma) = \left[I_{u_t} \otimes \left(\frac{W'X}{n} \left(\frac{X'X}{n} \right)^{-1} \frac{X'W}{n} \right)^{-1} \frac{W'X}{n} \left(\frac{X'X}{n} \right)^{-1} \right] n^{-1/2} \text{vec}(X' \varepsilon). \quad (26)$$

$$n^{-1/2} \text{vec}(X' \varepsilon) \xrightarrow{d} N(0, D) \quad (27)$$

using an MDS central limit theorem (c.f. Hamilton, Proposition 7.9), because (u_t) is an i.i.d. process with a finite fourth moment. Finally, (26) and (27) above prove the claimed results from Slutsky's theorem. ■

Theorem 4.1: Note:

- (i) $\pi(\alpha) \geq 0$ for any α ;
- (ii) $\hat{\gamma} \xrightarrow{p} \gamma(\tilde{\alpha})$;
- (iii) We claim that $p \lim \pi(\hat{\alpha}) = 0$ if $\hat{\alpha}$ is a CMD estimator. If $p \lim \pi(\hat{\alpha}) > 0$, this contradicts with the claim that $\hat{\alpha}$ is an α which minimizes the continuous function $\pi(\alpha)$, because $p \lim \pi(\tilde{\alpha}) = 0$ from $p \lim [\hat{\gamma} - \gamma(\tilde{\alpha})] = 0$ using (ii);
- (iv) So we claim $\hat{\alpha} \xrightarrow{p} \tilde{\alpha}$. If it is not, then $\hat{\alpha} \xrightarrow{p} \bar{\alpha}$ where $\bar{\alpha} \neq \tilde{\alpha}$. In this case, $p \lim \pi(\hat{\alpha}) > 0$ because

$$\begin{aligned} p \lim \pi(\hat{\alpha}) &= p \lim \text{vec}[\hat{\gamma} - \gamma(\alpha)]' \text{vec}[\hat{\gamma} - \gamma(\alpha)] \\ &= p \lim \{ \text{vec}[\hat{\gamma} - \gamma(\tilde{\alpha})] + \nabla \text{vec}[\gamma(\alpha^*)](\hat{\alpha} - \tilde{\alpha}) \}' \{ \text{vec}[\hat{\gamma} - \gamma(\tilde{\alpha})] + \text{vec}[\nabla \gamma(\alpha^*)](\hat{\alpha} - \tilde{\alpha}) \} \\ &= p \lim (\hat{\alpha} - \tilde{\alpha})' \nabla \text{vec}[\gamma(\alpha^*)]' \nabla \text{vec}[\gamma(\alpha^*)](\hat{\alpha} - \tilde{\alpha}) \\ &= p \lim (\bar{\alpha} - \tilde{\alpha})' \nabla \text{vec}[\gamma(\alpha^*)]' \nabla \text{vec}[\gamma(\alpha^*)](\bar{\alpha} - \tilde{\alpha}) > 0 \end{aligned}$$

from Taylor's expansion for the second equality, from (ii) for the third equality, $\hat{\alpha} \xrightarrow{p} \bar{\alpha}$ for the fourth equality, and $\nabla \text{vec}[\gamma(\alpha^*)]' \nabla \text{vec}[\gamma(\alpha^*)]$ is positive definite because $\nabla \text{vec}[\gamma(\alpha^*)]$ has a full column rank from the assumption for the final inequality. Therefore, $\hat{\alpha}$ is not a CMD estimator from (iii). Consequently, if $\hat{\alpha}$ is a CMD estimator, then the claimed results hold. ■

Theorem 4.2: Note $\gamma(\alpha)$ is continuously differentiable in the neighborhood N of $\tilde{\alpha}$, where $G(\alpha)$ is continuous at $\tilde{\alpha}$. Further, note that $n^{1/2} \text{vec}[\hat{\gamma} - \gamma(\tilde{\alpha})] \xrightarrow{d} N(0, HDH')$ from Theorem 3.6. Therefore, the claimed result holds from Theorem 3.2 in Newey and McFadden (1994). ■

Corollary 4.3: $\hat{\Omega} \xrightarrow{p} \Omega$ and $\hat{\alpha} \xrightarrow{p} \alpha$ show the claimed result, because R_i is a continuous function of α . ■

Theorem 5.1: Note that

$$\begin{aligned} \hat{\psi}_j^g(i) - \psi_j^g(i) &= \frac{[(e_j' \Sigma \otimes I_k), (e_j' \Sigma \otimes \kappa_i) - \frac{1}{2} \sigma_{jj}^{-\frac{1}{2}} \psi_j^g(i) (e_j' \otimes e_j')]}{k \times k^2} \frac{1}{\sqrt{\sigma_{jj}}} \sqrt{n} \begin{bmatrix} \text{vec}(\hat{\kappa}_i - \kappa_i) \\ \text{vec}(\hat{\Sigma} - \Sigma) \end{bmatrix}_{k^2 \times 1} + o_p(1) \\ &= \frac{[(e_j' \Sigma \otimes I_k), (e_j' \Sigma \otimes \kappa_i) - \frac{1}{2} \sigma_{jj}^{-\frac{1}{2}} \psi_j^g(i) (e_j' \otimes e_j')]}{k \times k^2} \frac{1}{\sqrt{\sigma_{jj}}} \times \end{aligned}$$

$$\begin{pmatrix} -(\bar{\varepsilon}' \otimes \bar{\varepsilon}') \left[\sum_{\ell=0}^{i-1} (A')^{i-1-\ell} \otimes A^\ell \right] \frac{\partial \text{vec}(A)}{\partial \alpha_v'} (G_v' G_v)^{-1} G_v' & 0 \\ (\Lambda(G_v' G_v)^{-1} G_v' + [\sum_{\ell=0}^{h-1} (R_\ell' \otimes R_\ell')])^{-1} [(I_k \otimes p \lim \varepsilon' W / n) + (p \lim \varepsilon' W / n \otimes I_k) K] & [\sum_{\ell=0}^{h-1} (R_\ell' \otimes R_\ell')^{-1}] \end{pmatrix} \begin{pmatrix} \sqrt{n} \text{vec}(\hat{\gamma} - \gamma) \\ (1/\sqrt{n}) \text{vec}(\varepsilon' \varepsilon - n\Omega) \end{pmatrix} + o_p(1) \quad (28)$$

from Pesaran and Shin (1998; Appendix A.14) for the first equality, and following results (a) and (b) for the second equality;

(a)

$$\begin{aligned} \sqrt{n} \text{vec}(\hat{\kappa}_i - \kappa_i) &= \frac{\partial \text{vec}(\kappa_i)}{\partial \alpha_v'} \sqrt{n}(\hat{\alpha}_v - \alpha_v) + o_p(1) \\ &= -(\bar{\varepsilon}' \otimes \bar{\varepsilon}') \left[\sum_{j=0}^i (A')^{i-1-j} \otimes A^j \right] \frac{\partial \text{vec}(A)}{\partial \alpha_v'} (G_v' G_v)^{-1} G_v' \sqrt{n} \text{vec}[(\hat{\gamma} - \gamma)] + o_p(1) \end{aligned}$$

by the Taylor series expansion for the first equality; and using

$$\frac{\partial \text{vec}(\kappa_i)}{\partial \alpha_v'} = (\bar{\varepsilon}' \otimes \bar{\varepsilon}') \frac{\partial \text{vec}(A^i)}{\partial \alpha_v'} = (\bar{\varepsilon}' \otimes \bar{\varepsilon}') \left[\sum_{j=0}^{i-1} (A')^{i-1-j} \otimes A^j \right] \frac{\partial \text{vec}(A)}{\partial \alpha_v'}$$

because $\text{vec}(\kappa_i) = (\bar{\varepsilon}' \otimes \bar{\varepsilon}') \text{vec}(A^i)$ from (25) and Lütkepohl (1993, p. 471) for differentiation; and expanding $\text{vec}(\hat{\gamma} - \gamma)$ in the first order condition of minimization problem (15) around α_v gives

$$\sqrt{n}(\hat{\alpha}_v - \alpha_v) = -(G_v' G_v)^{-1} G_v' \sqrt{n} \text{vec}[(\hat{\gamma} - \gamma)] + o_p(1), \quad (29)$$

for the second equality (c.f., Newey and McFadden, 1994; p. 2145).

(b)

$$\begin{aligned} \sqrt{n} \text{vec}(\hat{\Sigma} - \Sigma) &= \sqrt{n} \left(\left[\sum_{i=0}^{h-1} (R_i'(\hat{\alpha}) \otimes R_i(\hat{\alpha})) \right]^{-1} - \left[\sum_{i=0}^{h-1} (R_i' \otimes R_i') \right]^{-1} \right) \text{vec}(\hat{\Omega}) \\ &\quad + \left[\sum_{i=0}^{h-1} (R_i' \otimes R_i') \right]^{-1} \sqrt{n} \left[\text{vec}(\hat{\Omega}) - \text{vec}(\Omega) \right] \\ &= -\Lambda(G_v' G_v)^{-1} G_v' \sqrt{n} \text{vec}[(\hat{\gamma} - \gamma)] \\ &\quad + \left[\sum_{i=0}^{h-1} (R_i' \otimes R_i') \right]^{-1} \left(\frac{\text{vec}(\varepsilon' \varepsilon - n\Omega)}{\sqrt{n}} + [(I \otimes p \lim \varepsilon' W / n) + (p \lim \varepsilon' W / n \otimes I) K] \sqrt{n} \text{vec}[(\hat{\gamma} - \gamma)] \right) + o_p(1) \end{aligned}$$

from equation (19) for the first equality and following results (i) and (ii);

(i)

$$\sqrt{n} \left\{ \left[\sum_{i=0}^{h-1} (R_i'(\hat{\alpha}) \otimes R_i(\hat{\alpha})) \right]^{-1} - \left[\sum_{i=0}^{h-1} (R_i' \otimes R_i) \right]^{-1} \right\} \text{vec}(\hat{\Omega}) = \Lambda \sqrt{n}(\hat{\alpha}_v - \alpha_v) + o_p(1)$$

by the Taylor series expansion and using $\frac{\partial A^{-1}}{\partial x_j} = -A^{-1} \frac{\partial A}{\partial x_j} A^{-1}$ for a matrix A from Lütkepohl (1993, p. 471) where x_j is a j -th element of a vector α_v .

(ii)

$$\begin{aligned} \sqrt{n} \left[\text{vec}(\hat{\Omega}) - \text{vec}(\Omega) \right] &= \frac{\text{vec}(\varepsilon' \varepsilon - n\Omega)}{\sqrt{n}} + (I_k \otimes \varepsilon' W / n) \sqrt{n} \text{vec}[(\gamma - \hat{\gamma})] + \\ &(\varepsilon' W / n \otimes I_k) \sqrt{n} \text{vec}[(\gamma - \hat{\gamma})] + o_p(1) \end{aligned}$$

$$\text{because } \hat{\Omega} = \frac{\hat{\varepsilon}' \hat{\varepsilon}}{n} = \frac{[\varepsilon + W(\gamma - \hat{\gamma})]' [\varepsilon + W(\gamma - \hat{\gamma})]}{n} \quad \text{and} \quad \frac{(\gamma - \hat{\gamma})' W' W (\gamma - \hat{\gamma})}{\sqrt{n}} = o_p(1)$$

for the first equality where K is a communication matrix such that $\text{vec}[(\gamma - \hat{\gamma})]' = K \text{vec}[(\gamma - \hat{\gamma})]$.

Now we get the claimed result as

$$\hat{\psi}_j^g(i) - \psi_j^g(i) \xrightarrow{d} N(0, \Psi_j(i) \Gamma \Psi_j(i)')$$

from (26) and (28) using the central limit theorem for a martingale difference sequence (c.f. Hamilton, Proposition 7.9) and Slutsky's theorem because

$$n^{-1/2} \begin{pmatrix} \text{vec}(X' \varepsilon) \\ \text{vec}(\varepsilon' \varepsilon - n\Omega) \end{pmatrix} \xrightarrow{d} N(0, \Gamma),$$

$$\text{where } \Gamma \equiv E \left[\begin{pmatrix} \text{vec}(X_i \varepsilon_i') \\ \text{vec}[(\varepsilon_i \varepsilon_i') - \Omega] \end{pmatrix} \begin{pmatrix} \text{vec}(X_i \varepsilon_i') \\ \text{vec}[(\varepsilon_i \varepsilon_i') - \Omega] \end{pmatrix}' \right] \quad \text{using } E_{t-1} \begin{pmatrix} \text{vec}(X_t \varepsilon_t') \\ \text{vec}[(\varepsilon_t \varepsilon_t') - \Omega] \end{pmatrix} = 0.$$

■

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