The Korean Economic Review
Volume 32, Number 2, Winter 2016, 295-329.

Semi-parametric Method for Estimating Tail Related Risk Measures in the Stock Market

Hojin Lee*

The generalized Pareto distribution (GPD) approach for estimating the Value-at-Risk (VaR) and the expected shortfall (ES) is compared to other methods for evaluating extreme risk with normally distributed returns. When the market index returns have a fat-tailed distribution, the risk measures computed from the normal distribution underestimate the tail-related risk. We also compare the computation results of the VaR based on the GPD approximations to those based on the RiskMetrics methodology and GARCH model estimation. The estimates of the VaR are robust to a variety of threshold values. Contrary to this, the VaR values based on the RiskMetrics methodology and the GARCH model are extremely volatile. From a risk manager's perspective, it would be difficult to adjust capital requirement of a financial institution to conditional market risk. Due to concerns raised for practical and statistical reasons, we can conclude that the GPD method for measuring unconditional market risk is more appropriate for measuring and managing the tail-related risk.

JEL Classification: G15, F31, C46
Keywords: Generalized Extreme Value Distribution, Fat-tail Behavior, Value-at-Risk, Expected Shortfall, Generalized Pareto Distribution, Fisher-Tippett Theorem

I. Introduction

A traditional approach to measuring risk is based on the assumption of marginal and multivariate distribution of normality and linear dependency among the variables. Traditional methods derive the empirical density functions with observations heavily distributed around the center but sparsely distributed on both tails of the distribution. However, a large body of literature accumulates evidences of non-normality of the distribution and the time-varying correlation of portfolio

Received: Feb. 16, 2016. Revised: April 24, 2016. Accepted: June 2, 2016.

* Professor, Department of Business Administration, Myongji University, 34, Geobukgol-ro, Seodaemun-gu, Seoul 03674, Korea. e-mail: hlee07@mju.ac.kr. We thank the referees for very insightful comments on earlier drafts. The usual disclaimer applies.

returns. The extreme value theory (EVT) can be used in deriving the limiting distribution of fluctuating maxima or the excesses over a high threshold, so that we may fit the limiting distribution to the sparsely distributed observations in the tails. In light of the consideration, this paper investigates the limiting distribution of tails when traditional parametric and nonparametric estimation methodologies are inefficient due to scarce observations. The EVT estimator is used to estimate the tail probability and extreme quantile of the extreme random variable distribution that are required in measuring and managing the risk associated with extreme events.

The extant literature exploits the extreme value theory in finance and econometrics. Akgiray et al. (1988) analyze exchange rate returns and test for the infinite variance stable law hypothesis under the assumption that the outliers of the returns are distributed to the stable Paretian laws. Koedijk et al. (1990) employ extreme value theory to estimate the tail index and show that exchange rate returns have a fat-tailed distribution. Hols and de Vries (1991) employ nonparametric procedures based on order statistics to estimate the tail index and derive bounds on the exchange rate returns for very low probabilities. Jansen and de Vries (1991), pursuing the same strategy with extreme value theory, investigate the probability mass in the tails of stock returns and prove the existence of finite first and second moments. Loretan and Phillips (1994) estimate the tail index and determine the asymptotic distribution to test for the covariance stationarity of stock returns. Longin (1996) shows that an extreme movement of the U.S. market index has a Fréchet distribution. Kearns and Pagan (1997) present the simulation evidence on the need to allow for modeling dependence structure in the data when calculating risk with the significant probability of large deviations. Booth et al. (1997) use extreme value theory to extrapolate the probability of large deviations in stock index futures market and set the margin levels. Danielson and de Vries (1997) improve upon the efficiency of the tail index and quantile estimator by incorporating an estimate of the second order term of the tail expansion with high frequency foreign exchange rate data. McNeil and Saladin (1997) advocate the peaks over thresholds model to estimate high quantiles of loss severity distributions. McNeil (1998) compares the block maxima method for estimating the GEV with the excesses over threshold estimator via simulation. To be fair, his simulation experiment is based on the same number of observations. The block maxima procedure outperforms the excesses over threshold procedure on all criteria. Longin (2000) derives the limiting distribution of extreme returns based on extreme value theory and computes the Value-at-Risk (VaR). Dacorogna et al. (2001) shows that the high frequency data improve the efficiency of the tail shape estimators when the loss distribution exhibits power decline and that the heavy tails of the return distribution improve the portfolio diversification effect. With regard to modeling dependence structure in the data when calculating risk, Longin and Solnik (2001) and Ang and Chen (2002) investigate the effect of asymmetries in conditional correlations on risk management by differentiating downside correlations from upside correlations. Oh (2005) uses the two stage subsample bootstrapping method to select an optimal threshold and estimate the generalized Pareto distribution (GPD) model. It is found that the GPD model with an optimal threshold value produces superior VaR estimates to those from the generalized extreme value (GEV), the AR-GARCH and the RiskMetrics models. Tastan (2006) utilizes multivariate generalized autoregressive conditional heteroskedasticity model to explain the time varying nature of unconditional covariance between stock market index returns and nominal exchange rate changes. Hsu et al. (2012) integrate extreme value theory and a variety of copula distribution functions to search for an accurate fit to joint distribution of returns and compute VaR.

This paper aims to quantify the tail-fatness and evaluate the tail-related risk measures from the parameter estimates of the limiting distribution of the tails. We focus on modeling the heavy tails of the loss distribution. The EVT based approach offers the fully parametric methods of block maxima for estimating the tail of a loss distribution. It provides us with a rationale behind the statistical methodology with which we extrapolate beyond the range of the data. We are particularly interested in the excesses over a high threshold method. In the excesses over a high threshold model, we fit the GPD to the threshold excesses. Accordingly, the choice of an optimal threshold may be a critical issue. Goldie and Smith (1987), Hall (1990), Danielsson et al. (2001), Danielsson and de Vries (1998) and Oh (2005) employ a subsample bootstrap procedure to determine the optimal threshold value. McNeil and Saladin (1997), McNeil and Frey (2000) and Zivot and Wang (2006), on the other hand, suggest choosing the optimal number of extreme order statistics by the sample mean excess function plot. The main concern in implementing the maximum likelihood estimation of the GPD parameters is to choose a high threshold because a statistical procedure for selecting the optimal threshold remains arbitrary to some extent. We can minimize the bias of the GPD parameter estimates by choosing a high threshold. In theory, it is best to fit the GPD to the data solely pertained to the tail of the distribution and not included in the center of the distribution. At the same time, however, we want to reduce the variance of the parameter estimates by keeping the number of observations included in the tail shape and scale parameter estimation large enough to have a sufficient number of exceedances of the losses over a high threshold. In this paper, we follow the sample mean excess methodology to see if the sample mean excess plot can provide us with the appropriate answer to selecting the optimal threshold.

Our contributions to these findings are as follows. Firstly, we derive the GEV distribution of the index returns on four stock market indices from the block maxima. Armed with the parameter estimates of the GEV distribution, we compute the return level from the estimated model. In our context of block maxima, the return level indicates a level of return which is exceeded by a block maximum return with a certain level of probability and can be interpreted as a risk measure. We investigate the trade-off relationship between the bias and the efficiency of the MLE method in estimating the GEV distribution using block maxima. We choose the number of observations per block to have the annual block maxima, the semester block maxima, the quarterly block maxima, the monthly block maxima and the weekly block maxima, respectively. The tail shape parameter estimates of the market indices with a variety of block maxima are consistent for the KOSPI and the NIKKEI 225. For the S&P 500 and the FTSE, the estimates have a tendency to increase with the number of observations in the block. For most indices, the tail shape parameter estimates with the weekly block maxima have negative values with large standard errors. In sum, we find that the tail shape parameter estimates of the GEV distribution with the block maxima are robust to the choice of the number of observations per block.

Secondly, we approximate the EVT distribution of the excesses over a high threshold by the GPD methodology to obtain efficient parameter estimates of the model. We balance between minimizing the bias of the GPD parameter estimates by choosing a high threshold and reducing the variance of the parameter estimates by including a sufficient number of exceedances over a high threshold in the tail. We investigate the GPD specifications across a variety of threshold values. We set the initial value of the threshold as zero for the GPD model. We then increase the threshold value consecutively to reach the point where the model estimation contains less than 1% of the total observations. The tail shape parameter estimates of the GPD are relatively constant across a variety of threshold values. We agree that the two stage subsample bootstrapping method to select an optimal threshold as in Danielsson and de Vries (1998) and Oh (2005) is statistically rigorous and efficient. However, we find that the sample mean excess function methodology to choose a high threshold can also be useful.

Thirdly, we use a semi-parametric method for estimating the market risk from an unconditional return distribution. We combine a parametric assessment of the VaR with the fitted tail distribution to form a semi-parametric evaluation of the VaR. By sampling from the tail of the distribution, the level of statistical precision is elevated in evaluating the VaR. We employ the GPD model to estimate the tail probability and the extreme quantile of the distribution that are required in measuring and managing the risk associated with extreme events. We compare the computation results of the VaR based on the GPD approximations to those based on the RiskMetrics methodology and the GARCH model estimation. The robustness test results of the VaR estimates based on the GPD specification are established. Contrary to this, the VaR values based on the RiskMetrics methodology and the GARCH model are extremely volatile. We conclude that the GPD method for measuring unconditional market risk is appropriate for measuring and managing the tail-related risk.

The rest of the paper is organized as follows. Section 2 discusses the limiting distribution of extreme random variables. We discuss the extreme value theory method for estimating the limiting distribution from the block maxima data. We then present an alternative extreme value theory procedure for estimating the limiting distribution from the excesses over a high threshold. We also discuss a parametric assessment of the risk measures with the fitted tail distribution to form a semi-parametric evaluation of the VaR and the ES. In section 3, we present the empirical results. Section 4 concludes our discussion.

II. The Model

2.1. The GEV Distribution

We discuss the distribution of the extreme losses on the stock market index. The probability mass in the tails is used more effectively in extrapolating from historical events to unprecedented levels by discarding the observations in the center of the distribution. As the first approach to extrapolating the empirical distribution of the maximum order statistic, we derive the limiting distribution of location-scale adjusted maxima.

From the *iid* sequence of extreme random variable $X_i, i = 1, 2, \ldots T$ with a cumulative distribution function F, we define X_i as daily observations of the negative log return on the stock market index. We define $M_n^{(j)} = max(X_1^j, \cdots, X_n^j)$, $j = 1, \cdots, m$ and $n = T / m$ as the block maxima from m subsamples. The limit law for fluctuating maxima due to the Fisher and Tippett (1928) theorem states that the normalized maxima is distributed to the GEV distribution with the tail shape parameter ξ and the location-scale parameter μ_n and σ_n as in equation (1) (Embrechts et al., 1997; Coles, 2001; Zivot and Wang, 2006). The 1-month period is chosen for the block size n, so we have $n = \frac{7,352}{350} = 21$ observations in each block and 350 monthly block maxima from the sample.

The form of the GEV $H_\xi(x)$ is:

$$H_\xi(x) \begin{cases} \exp\{-(1+\xi x)^{-\frac{1}{\xi}}\} & \xi \neq 0 \\ \exp\{-\exp(-x)\} & \xi = 0 \end{cases}$$
$$\text{where, } 1 + \xi x > 0 \,. \tag{1}$$

The tail shape parameter ξ differentiates the limiting distribution of block maxima, so that $\xi > 0$ corresponds to the Frechét distribution, $\xi = 0$ to the Gumbel distribution, and $\xi < 0$ to the Weibull distribution.

The form of the location-scale normalized GEV distribution $H_{\xi,\mu,\sigma}$ is:

$$H_{\xi}(z) = H_{\xi}\left(\frac{x-\mu}{\sigma}\right) = H_{\xi,\mu,\sigma}(x), \tag{2}$$

where the extreme random variable x is normalized by the location-scale parameters $\mu \in R$, $\sigma > 0$. The Fisher-Tippet theorem states that there exist norming constants μ and $\sigma > 0$ such that the standardized maxima tend to the GEV distribution.

Out-of-sample forecast of the probability of observing a new record negative return at the 1-period horizon is made from the estimates of the tail shape parameter and the norming constants. We also forecast the loss quantiles at different risk levels which is exceeded by the maximum loss at the 1-period horizon with a given level of probability.

2.2. The GPD Function

Since the block maxima are selected from m disjoint blocks of n observations and used for the GEV distribution estimation, most of the information contained in the full sample of data is ignored. When we aggregate a set of data by calendar months into m blocks of approximately n days in each block, we discard $n-1$ observations in each block which may cause the efficiency problem of the block maxima estimation of the GEV distribution parameters. As an alternative approach to estimating the tails of the loss distribution, we use the limiting distribution of threshold excesses.

We fix a high threshold u and denote a random number n_u as the number of exceedances of $X_i (i = 1,\ldots,n_u)$ over the threshold u. According to Embrechts et. al. (1997), McNeil and Saladin (1997), McNeil and Frey (2000) and Zivot and Wang (2006), it is possible to approximate the distribution of the excesses over a high threshold u by a generalized Pareto distribution (GPD) (3) with a positive tail shape parameter ξ and a positive measurable scale function $\beta(u)$

$$G_{\xi,\beta(u)}(x) = \begin{cases} 1-\left(1+\dfrac{\xi x}{\beta(u)}\right)^{-\frac{1}{\xi}} & \text{for } \xi \neq 0 \\ 1-\exp\left(-\dfrac{x}{\beta(u)}\right) & \text{for } \xi = 0 \end{cases} \tag{3}$$

defined for $x \geq 0$ when $\xi \geq 0$ and $0 \leq x \leq -\beta(u)/\xi$ when $\xi < 0$.

The GPD approach is operable with an optimal threshold over which we have enough observations to obtain efficient parameter estimates of the model. At the same time, care must be taken to select a high threshold to minimize the bias of the

GPD parameter estimates. From a statistical viewpoint, the impediment in implementing the maximum likelihood estimation of the GPD parameters is to choose a high threshold because a statistical procedure for selecting the optimal threshold remains arbitrary to some extent. We can minimize the bias of the GPD parameter estimates by choosing a high threshold. In theory, it is best to fit the GPD to the data solely pertained to the tail of the distribution and not included in the center of the distribution. At the same time, however, we want to reduce the variance of the parameter estimates by keeping n_u large enough to have a sufficient number of exceedances of X over a high threshold. McNeil and Saladin (1997) suggest the sample mean excess function method for selecting the optimal number of extreme order statistics to be determined by the start of the tail u.

The form of the sample mean excess function $e_n(u)$ is:

$$e_n(u) = \frac{1}{n_u} \sum_{i=1}^{n_u} (X_i - u) \tag{4}$$

where $X_i (i = 1, \ldots, n_u)$ are the values of X_i such that $X_i > u$.

We follow the interpretation of the mean excess function plot due to McNeil and Saladin (1997) and Zivot and Wang (2006), among others. We base our decision on the GPD with a positive scale function and a tail shape parameter by the linearity of the sample mean excess function plot. That is, we may choose a threshold value of u over which the excesses are distributed to a GPD if the sample mean excess function plot is linear with positive slope. This can be seen from the form of the mean excess function equation that the sample mean excess is linearly correlated with a threshold value of u for a fixed value of ξ as follows.

$$e(u) = E[X - u \mid X > u] = \frac{\beta(u_0) + \xi u}{1 - \xi}$$

We define the extreme order statistics above a high threshold u from the first to the n_u^{th} as $X_{(1)}, \ldots, X_{(n_u)}$ and the threshold excesses as $y_i = X_{(i)} - u$ for $i = 1, \ldots, n_u$. The threshold excesses $[y_1, \ldots, y_{n_u}]$ are distributed to a GPD with unknown parameters ξ and $\beta(u)$. For a sufficiently high threshold u, the form of the tail distribution or the distribution of X_i for which $X_i - u > 0$ is:

$$1 - F(x) = (1 - F(u))(1 - F_u(y)) \approx (1 - F(u))(1 - G_{\xi, \beta(u)}(y)) \tag{5}$$

where we define the excess distribution above the threshold u as $F_u(y)$, and for a sufficiently high threshold u, $F_u(y) \approx G_{\xi, \beta(u)}(y)$.

Since the number of exceedances of X_i over a high threshold u in the tail is n_u, $1 - F(u)$ in equation (5) may be estimated non-parametrically using the empirical CDF as $\frac{n_u}{n}$. If we rearrange equation (5), the tail estimator is obtained as follows:

$$\hat{F}(x) = 1 - \frac{n_u}{n}\left(1 + \hat{\xi} \cdot \frac{x-u}{\hat{\beta}(u)}\right)^{-\frac{1}{\hat{\xi}}} \qquad \text{for} \quad x > u \tag{6}$$

where $\hat{\xi}$ and $\hat{\beta}(u)$ denote the maximum likelihood estimates of ξ and $\beta(u)$, respectively. The semi-parametric estimation of the tails of the loss distribution is obtained using the non-parametric estimate of the random proportion of the data and the GPD function of estimating the tails of the loss distribution.

2.3. Market Risk Measures

The aim of the VaR analysis lies in the accurate forecasting of extreme events. Danielsson and de Vries (1997) propose a semi-parametric method for estimating the market risk from an unconditional return distribution. McNeil and Frey (2000), on the other hand, suggest applying an extreme value theory to estimate the tail of the fitted GARCH residual which is assumed to be a conditional return distribution. However, GARCH models are known to have poor tail properties and are not designed to evaluate the VaR. From a risk manager's perspective, it would be difficult to adjust the capital requirement of a financial institution to conditional market risk. The use of a conditional return distribution also raises concerns related to constructing conditional variance-covariance measures when a portfolio is composed of a large number of assets. Due to the concerns raised for practical and statistical reasons, we use a tail-focused semi-parametric method for measuring unconditional market risk.

In this paper, we combine a parametric assessment of the VaR with the fitted tail distribution to form a semi-parametric evaluation of the VaR. By sampling from the tail of the distribution, the level of statistical precision is elevated in evaluating the VaR. Depending on the characterization of the probability distribution function $F(\cdot)$, we use different methods for evaluation of the VaR. In a parametric assessment of VaR, for $0 < \alpha < 1$, a $100 \cdot \alpha\%$ quantile of the distribution function F is the value q_α which satisfies

$$q_\alpha = \inf\{x \in R : F(x) \geq \alpha\},$$

For $q_\alpha > F(u)$, the form of the EVT estimate of VaR is calculated by inverting the tail estimator in equation (6) to get

$$\widehat{VaR_\alpha} = u + \frac{\hat{\beta}(u)}{\hat{\xi}}\left\{\left[\frac{n}{n_u}(1-q_\alpha)\right]^{-\hat{\xi}} - 1\right\} \tag{7}$$

If the loss distribution function has fatter tails than the normal distribution, the $100 \cdot \alpha\%$ quantile of the normal distribution could be misleading and underestimate the EVT estimate of the risk measures. When the unconditional return distribution has a heavy tail on the downside risk, the estimates of the extreme quantile based on the excesses over a high threshold procedure are more accurate.

As another frequently examined risk measure, the ES is calculated as the conditional expectation of the losses X given that X is greater than VaR_α. Also, the form of the EVT estimate of the ES_α is related to VaR_α via

$$ES_\alpha = VaR_\alpha + E[X - VaR_\alpha \mid X > VaR_\alpha] \tag{8}$$

We compute the ES_α as the conditional expectation of the threshold excesses $F_{VaR_\alpha}(y)$ given that X is greater than VaR_α. The GPD approximation to $F_{VaR_\alpha}(y)$ has the shape parameter ξ and the scale parameter $\beta(u) + \xi(VaR_\alpha - u)$. Consequently, using (7)

$$E[X - VaR_\alpha \mid X > VaR_\alpha] = \frac{\beta(u) + \xi(VaR_\alpha - u)}{1 - \xi} \tag{9}$$

provided $\xi < 1$. According to Zivot and Wang (2006), the GPD approximation to ES_q is derived by combining (9) with (7) and substituting into (8).

$$\widehat{ES}_\alpha = \frac{\widehat{VaR}_\alpha}{1 - \hat{\xi}} + \frac{\hat{\beta}(u) - \hat{\xi} u}{1 - \hat{\xi}} \tag{10}$$

III. Empirical Results

3.1. Data

Figure 1 shows the daily closing prices and the continuously compounded returns on the KOSPI, S&P 500, FTSE, NIKKEI during the period from January 1985 to February 2014 (7,352 observations). From the descriptive statistics reported in Table 1, while the distribution of the index returns on the S&P 500, FTSE and NIKKEI have long tail to the left, the distribution of the KOSPI returns has long tail to the right. The return distributions have fatter tails than the normal distribution in all four stock market indices. The Jarque-Bera statistics for normality test result in rejecting the null hypotheses in all cases under the 1% significance level. The mean values of the continuously compounded returns are less than a tenth of a percentage and follow the martingale process.

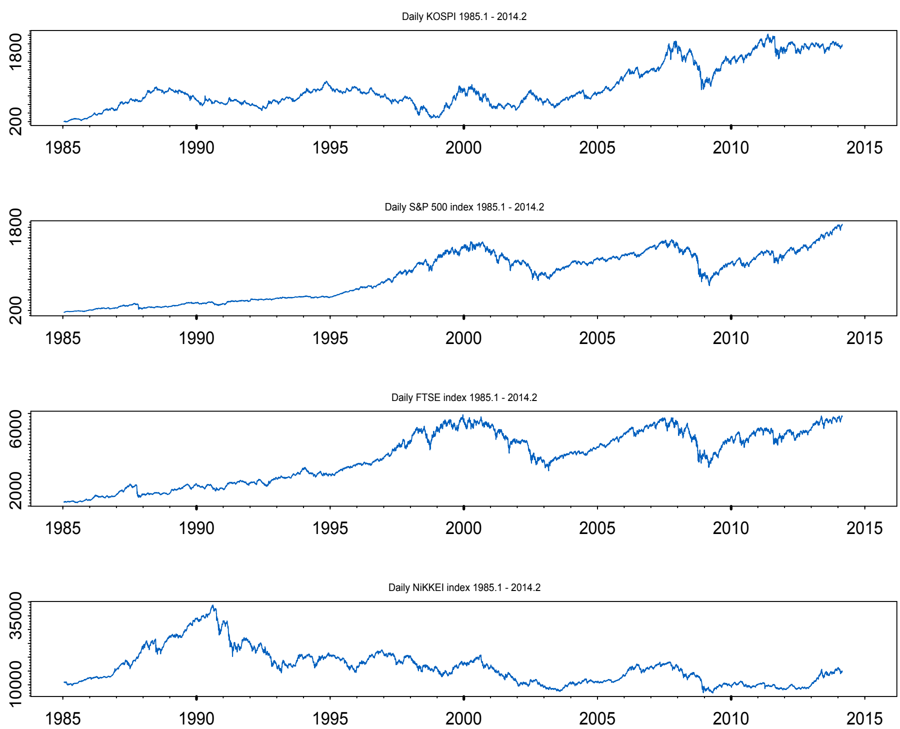
[Table 1] Statistical summary of index returns

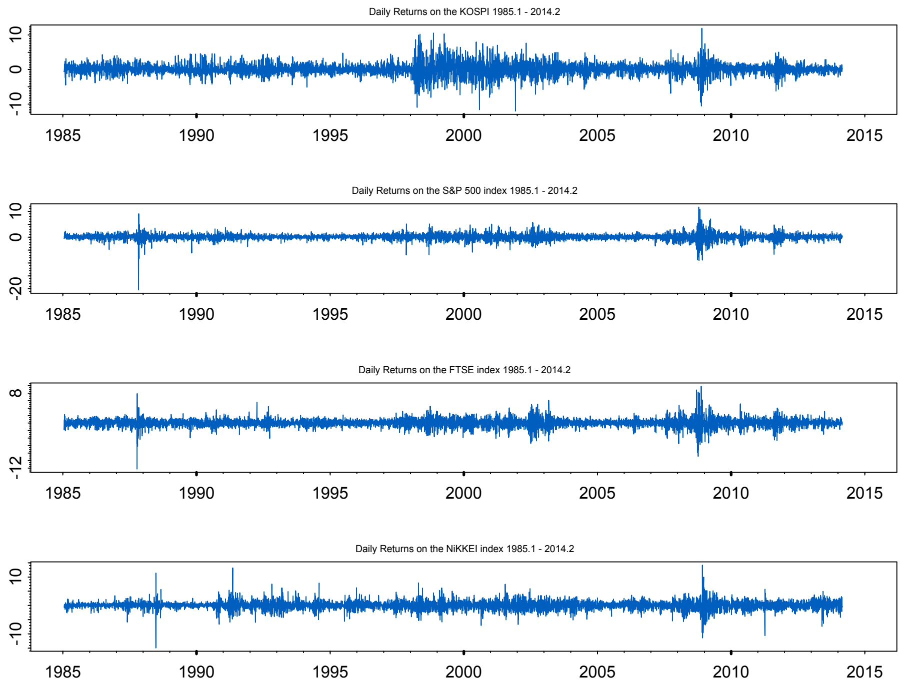
(%)

	Minimum	25% quantile	Median	75% quantile	Maximum
KOSPI	-12.02	-0.77	0.04	0.85	11.95
S&P 500	-20.47	-0.46	0.06	0.57	11.58
FTSE	-12.22	-0.53	0.06	0.63	9.84
NIKKEI	-14.90	-0.70	0.04	0.75	14.15
	Mean	Standard deviation	Skewness	Kurtosis	Jarque-Bera p-value
KOSPI	0.04	1.68	0.04	7.92	0.00
S&P 500	0.04	1.16	-0.84	24.34	0.00
FTSE	0.03	1.11	-0.21	11.09	0.00
NIKKEI	0.01	1.47	-0.08	10.75	0.00

Note: The table reports descriptive statistics of the four index returns. The Jarque-Bera test for normality reports the p-values.

[Figure 1] Daily prices and log returns on the stock market index from 1985.1 to 2014.2





3.2. The GEV Distribution Estimation

We fit the GEV distribution in equation (2) to the m independent realizations of the block maxima random variable $\{M_n^{(1)},\ldots,M_n^{(m)}\}$. The 1-month period is chosen for the block size n, so we have $n=\frac{7,352}{350}=21$ observations in each block and 350 monthly block maxima from the sample. It is important to balance between having a sufficiently large number of observations n per block to correct for the bias of the MLE and not having too meagre number of maxima m to increase the efficiency of the MLE. The estimation results are provided in Table 2. Although the main concerns of the study are the calculations of the tail probabilities and the extreme quantiles, we also put our emphasis on the estimation of the parameters of the limiting distribution of extreme random variables. Since the influential works of Koedijk et al. (1990), Hols and de Vries (1991), Jansen and de Vries (1991), Loretan and Phillips (1994) and Longin (1996), the tail index estimate of ξ is used to determine the asymptotic distribution of financial time series and to test for the existence of finite moments. What we can read off from the results in Table 2 is that the tail index estimates of ξ in four stock market indices are less than 0.25, indicating a finite fourth moment. When $\xi \geq 0$, the GEV distribution of maxima belongs to the Frechét distribution.

[Table 2] GEV distribution fit to monthly block maxima

KOSPI	Value	Standard Error	t-ratio
ξ	0.22	0.05	4.77
σ	1.03	0.05	20.28
μ	1.95	0.06	31.04
S&P 500	Value	Standard Error	t-ratio
ξ	0.25	0.04	5.54
σ	0.72	0.04	20.19
μ	1.32	0.04	29.87
FTSE	Value	Standard Error	t-ratio
ξ	0.22	0.04	4.85
σ	0.63	0.03	20.50
μ	1.32	0.04	34.27
NIKKEI	Value	Standard Error	t-ratio
ξ	0.13	0.04	3.13
σ	0.99	0.05	21.61
μ	1.80	0.06	29.97

Note: The table reports the estimation results of the distribution (2), $H_{\xi,\mu,\sigma}(\frac{x-\mu_n}{\sigma_n}) = H_{\xi,\mu_n,\sigma_n}(x)$ with the Newton-Raphson algorithm for numerical maximization.

[Table 3] Extreme loss probability calculation from the GEV fit to monthly block maxima

	Monthly maximum loss (%)	Probability (%)
KOSPI	12.02	0.0054
S&P 500	20.47	0.0003
FTSE	12.22	0.0008
NIKKEI	14.90	0.0004

Note: The table reports the probability of observing an unprecedented minimal return (the monthly maximum loss) during the first out-of-sample period. The sample is composed of 350-month of data. We assume there are twenty-one trading days in each month. The subscript 21 stands for the number of days in each monthly block. The superscript 350 represents the number of months in the sample. The probabilities in the third column are the computation results of the following:
$\Pr(M_{21}^{(351)} > \max(M_{21}^{(1)},\ldots,M_{21}^{(350)})) = 1 - H_{\hat{\xi},\hat{\mu},\hat{\sigma}}$ (monthly maximum loss).

As an application of the GEV distribution parameter estimates, the out-of-sample forecast of the probability of a maximum loss at the 1-period horizon is made from the results in Table 2 and shown in Table 3. Since the maximum of the monthly block maxima is 12.02% for the KOSPI, the out-of-sample forecast of the probability of a maximum loss at the 1-period horizon is calculated as

$\Pr(M_{21}^{(351)} > \max(M_{21}^{(1)},\ldots,M_{21}^{(350)})) = 1 - H_{\hat{\xi},\hat{\mu},\hat{\sigma}}(12.02) = 0.0054\%$. That is, the probability that we observe a new record loss on the KOSPI index during the first out-of-sample period is 0.0054%.

Likewise, we construct a one-step-ahead forecast of the probability of a new record maximum loss using the quarterly block maxima data and report the results in Table 5, along with the GEV distribution estimation results in Table 4. The tail shape parameter estimates for the market indices with the quarterly block maxima hover around 0.17 - 0.41, and are slightly greater than the estimates with the monthly block maxima. The probability that the next quarter's maximum loss on the index is a new record minimal return with the quarterly block maxima is 0.0165% for the KOSPI. This probability is in line with the corresponding probability that we calculate from the monthly block maxima.

[Table 4] GEV distribution fit to quarterly block maxima

KOSPI	Value	Standard Error	t-ratio
ξ	0.30	0.09	3.30
σ	1.15	0.11	10.93
μ	2.67	0.12	21.51
S&P 500	Value	Standard Error	t-ratio
ξ	0.36	0.09	4.15
σ	0.79	0.07	10.75
μ	1.86	0.08	22.12
FTSE	Value	Standard Error	t-ratio
ξ	0.41	0.10	4.06
σ	0.60	0.06	10.10
μ	1.77	0.07	26.96
NIKKEI	Value	Standard Error	t-ratio
ξ	0.17	0.07	2.46
σ	1.14	0.09	12.37
μ	2.58	0.12	21.72

Note: The table reports the estimation results of the distribution (2), $H_{\xi,\mu,\sigma}(\frac{x-\mu_n}{\sigma_n}) = H_{\xi,\mu_n,\sigma_n}(x)$ with the Newton-Raphson algorithm for numerical maximization.

[Table 5] Extreme loss probability calculation from the GEV fit to quarterly block maxima

	Quarterly maximum loss (%)	Probability (%)
KOSPI	12.02	0.0165
S&P 500	20.47	0.0020
FTSE	12.22	0.0062
NIKKEI	14.90	0.0022

Note: The table reports the probability of observing an unprecedented minimal return (the monthly maximum loss) during the next quarter. The sample is composed of 117-quarter of data. We assume there are twenty-one trading days in each month. The subscript 63 stands for the number of days in each quarterly block. The superscript 117 represents the number of quarters in the sample. The probabilities in the third column are the computation results of the following:
$\Pr(M_{63}^{(118)} > \max(M_{63}^{(1)},\ldots,M_{63}^{(117)})) = 1 - H_{\hat{\xi},\hat{\mu},\hat{\sigma}}$ (monthly maximum loss).

We additionally investigate the trade-off relationship between the bias and the efficiency of the MLE method in estimating the GEV distribution using block maxima. We choose the number of observations per block to have the annual block maxima, the semester block maxima and the weekly block maxima, respectively. The number of blocks for each block maxima is 30, 59 and 2,127. The estimation results are presented in Tables 6, 7 and 8. The tail shape parameter estimates of the market indices with a variety of block maxima are consistent for the KOSPI (0.22 - 0.32) and the NIKKEI 225 (0.13 - 0.17). For the S&P 500 and the FTSE, those hover around 0.22 - 0.46 and have a tendency to increase with the number of observations in the block. For most indices, the tail shape parameter estimates with the weekly block maxima have negative values with large standard errors. In sum, we find that the tail shape parameter estimates of the GEV distribution with block maxima are robust to the choice of the number of observations per block.

[Table 6] GEV distribution fit to annual block maxima

KOSPI	Value	Standard Error	t-ratio
ξ	0.29	0.18	1.63
σ	1.56	0.28	5.65
μ	3.81	0.33	11.53
S&P 500	Value	Standard Error	t-ratio
ξ	0.51	0.25	2.03
σ	1.35	0.29	4.61
μ	2.71	0.31	8.75
FTSE	Value	Standard Error	t-ratio

ξ	0.43	0.21	2.01
σ	0.89	0.18	5.01
μ	2.61	0.20	13.33
NIKKEI	Value	Standard Error	t-ratio
ξ	0.14	0.15	0.97
σ	1.70	0.27	6.25
μ	4.33	0.35	12.26

Note: The table reports the estimation results of the distribution (2), $H_{\xi,\mu,\sigma}(\frac{x-\mu_n}{\sigma_n}) = H_{\xi,\mu_n,\sigma_n}(x)$ with the Newton-Raphson algorithm for numerical maximization.

[Table 7] GEV distribution fit to semester block maxima

KOSPI	Value	Standard Error	t-ratio
ξ	0.32	0.13	2.47
σ	1.29	0.17	7.62
μ	3.15	0.20	16.03
S&P 500	Value	Standard Error	t-ratio
ξ	0.46	0.14	3.23
σ	0.94	0.13	7.06
μ	2.18	0.14	15.17
FTSE	Value	Standard Error	t-ratio
ξ	0.46	0.15	3.04
σ	0.71	0.10	6.90
μ	2.07	0.11	18.77
NIKKEI	Value	Standard Error	t-ratio
ξ	0.13	0.10	1.37
σ	1.47	0.16	9.01
μ	3.35	0.21	15.62

Note: The table reports the estimation results of the distribution (2), $H_{\xi,\mu,\sigma}(\frac{x-\mu_n}{\sigma_n}) = H_{\xi,\mu_n,\sigma_n}(x)$ with the Newton-Raphson algorithm for numerical maximization.

[Table 8] GEV distribution fit to weekly block maxima

KOSPI	Value	Standard Error	t-ratio
ξ	-0.02	0.01	-2.11
σ	1.12	0.02	62.90
μ	0.67	0.03	25.65
S&P 500	Value	Standard Error	t-ratio
ξ	0.04	0.01	4.67

	Value	Standard Error	t-ratio
σ	0.71	0.01	61.23
μ	0.43	0.02	26.20
FTSE	Value	Standard Error	t-ratio
ξ	-0.01	0.01	-1.64
σ	0.75	0.01	63.48
μ	0.47	0.02	26.96
NIKKEI	Value	Standard Error	t-ratio
ξ	-0.02	0.01	-1.88
σ	0.99	0.02	63.10
μ	0.64	0.02	27.51

Note: The table reports the estimation results of the distribution (2), $H_{\xi,\mu,\sigma}(\frac{x-\mu_n}{\sigma_n}) = H_{\xi,\mu_n,\sigma_n}(x)$ with the Newton-Raphson algorithm for numerical maximization.

We compute an alternative risk measure, the so-called return level from the estimated GEV distribution. The form of the k-year return level $R_{n,k}$ is:

$$\Pr\{M_n > R_{n,k}\} = \frac{1}{k} \quad \text{for} \quad k > 1$$

We expect the return level to be exceeded in one period out of every k periods. In our context of block maxima, the return level indicates a level of return which is exceeded by a block maximum return with the probability of $1/k$, on average. If we assume that the block maxima are distributed to the GEV distribution, then the estimate of the return level is obtained from the inverse of the GEV distribution. The form of the return level expressed in terms of MLEs of ξ, μ and σ is:

$$\hat{R}_{n,k} = H^{-1}_{\xi,\mu,\sigma}\left(1-\frac{1}{k}\right) = \hat{\mu} - \frac{\hat{\sigma}}{\hat{\xi}}\left\{1-\left[-\log\left(1-\frac{1}{k}\right)\right]^{-\hat{\xi}}\right\}, \quad \text{for} \quad \xi \neq 0$$

Using $\Pr\{M_n < x\} \approx H_{\xi,\mu,\sigma}(\frac{x-\mu_n}{\sigma_n}) = H_{\xi,\mu_n,\sigma_n}(x) = F^n(x)$, for iid losses X with distribution function F, $H_{\xi,\mu,\sigma} \approx F^n$ so that

$$F(R_{n,k}) = \Pr(M_n \leq R_{n,k}) \approx \left(1-\frac{1}{k}\right)^{\frac{1}{n}} \tag{11}$$

From equation (11), we can interpret the return level $R_{n,k}$ as the $(1-\frac{1}{k})^{1/n}$ quantile of the loss distribution function F for iid losses. That is, the 40-year return level from the monthly maxima $R_{21,480}$ is the 99.99% quantile and the 40-year return level from the quarterly block maxima $R_{63,160}$ is the 99.99% quantile of the loss distribution. Given the parameter estimates of the GEV distribution, the 40-year return level of the GEV distribution function is computed and reported in Table 9 and Table 10. The return level estimate of 15.53% for the KOSPI returns means that the monthly maximum loss observed during a period of one month will exceed 15.53% only in one month out of every 40 years, on average. From the point of view of a risk manager, it seems that the KOSPI is riskier than any other market indices.

[Table 9] The 40-year return level of the loss distribution using monthly block maxima

	40-year return level	95% Lower bound	95% Upper bound
KOSPI	15.53	12.01	21.87
S&P 500	11.92	9.22	16.95
FTSE	9.51	7.48	13.20
NIKKEI	11.16	9.15	14.70

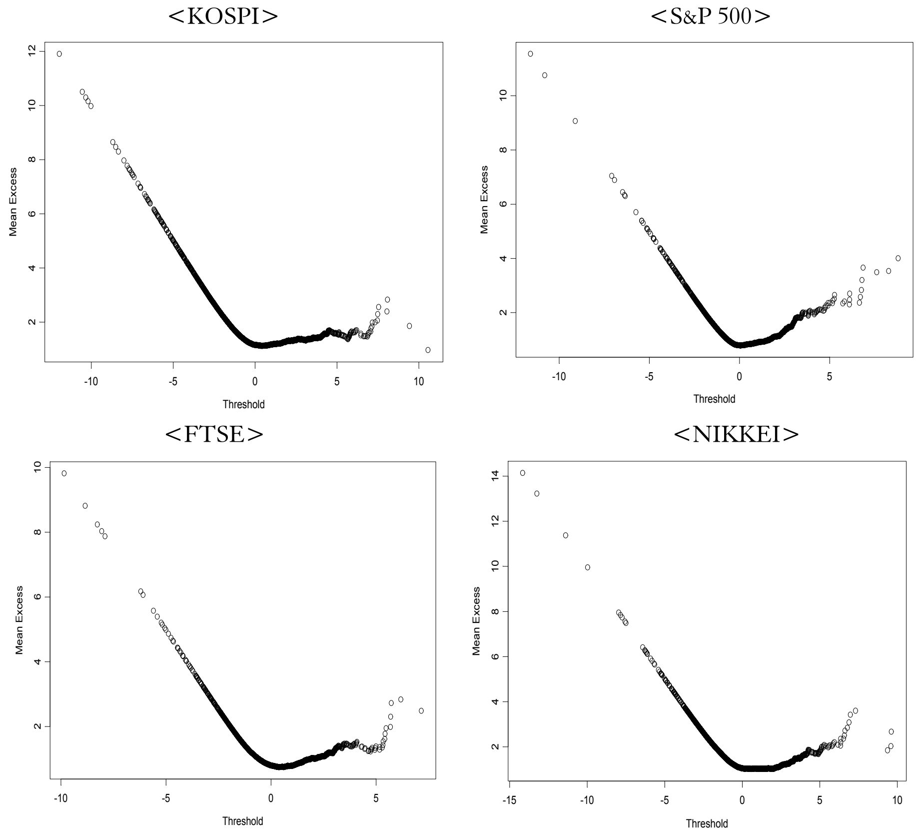
[Table 10] The 40-year return level of the loss distribution using quarterly block maxima

	99.9% Quantile	95% Lower bound	95% Upper bound
KOSPI	16.57	11.32	30.21
S&P 500	13.39	9.15	25.00
FTSE	12.15	7.64	24.98
NIKKEI	11.82	9.19	17.85

3.3. The GPD Function Estimation

We estimate the GPD parameters with excesses over a high threshold. The GPD approach is operable with an optimal threshold over which we have enough observations to obtain efficient parameter estimates of the model. At the same time, care must be taken to select a high threshold to minimize the bias of the GPD parameter estimates. We can minimize the bias of the GPD parameter estimates by choosing a high threshold. In theory, it is best to fit the GPD to the data solely pertained to the tail of the distribution and not included in the center of the distribution. At the same time, however, we want to reduce the variance of the parameter estimates by keeping n_u large enough to have a sufficient number of exceedances of X over a high threshold.

[Figure 2] Mean excess function plot



For locating the threshold value, we use the sample mean excess function plot in Figure 2. We have chosen the threshold value of one from which the mean excess function plot shows a straight line with positive slope. This suggests that the excesses over a threshold $u=1$ have Pareto distributed heavy tails. We choose a high threshold $u=1$ and denote the exceedances of $X_1,\ldots,X_{n_u}$ above u as the excess distribution of X over a high threshold. By selecting $u=1$, we include 20% of the total of 7,351 observations of the KOSPI data to fit the GPD. The GPD functions for the S&P 500 index, the FTSE and the NIKKEI 225 are fit to 12%, 13% and 19% of the observations of the data, respectively.

We report the GPD estimation results in Table 11 and Table 12. The maximum likelihood estimates for ξ, $\beta(1)$ and the asymptotic standard errors with the threshold $u=1$ are presented in Table 11. The tail shape parameter estimate of ξ for each index returns ranges from 0.07 to 0.17 which may be interpreted as having a heavy tail. We also report the GPD parameter estimation results with extreme 5% of the observations in Table 12. We choose the upper and the lower threshold value as -2.55 and 2.55 so that the tails of the GPD function on both extremes include 5%

of the observations of the KOSPI returns. While the number of exceedances in the lower tail above a threshold value of -2.55 accounts for 5.27% of the data, the number in the upper tail above a threshold value of 2.55 contains 5.03% of the data indicating the tail shape asymmetry. Compared with the results in Table 11, the tail shape parameter estimates of the KOSPI and the FTSE index returns in Table 12 do not have much difference in values. For the S&P 500 index and the NIKKEI returns, we observe a slight increase in the tail shape parameter estimates. We also confirm these results in Figure 3.

[Table 11] GPD $G_{\xi,\beta(u)}$ estimation results with a threshold value of $u=1$

	Value	Standard Error	t-ratio
KOSPI			
ξ	0.09	0.03	3.12
β	1.06	0.04	25.79
S&P 500			
ξ	0.17	0.04	4.83
β	0.73	0.04	20.62
FTSE			
ξ	0.17	0.04	4.57
β	0.65	0.03	20.16
NIKKEI			
ξ	0.07	0.03	2.93
β	0.94	0.03	27.02

Note: The table reports the maximum likelihood estimation results of the GPD parameters which are used in constructing the semi-parametric estimation of the tail estimator.

[Table 12] GPD $G_{\xi,\beta(u)}$ estimation results with extreme 5% of the observations

	Value	Standard Error	t-ratio
KOSPI			
u	2.55		
ξ	0.07	0.06	1.26
β	1.24	0.09	13.08
S&P 500			
u	1.72		
ξ	0.27	0.06	4.21
β	0.74	0.06	12.26

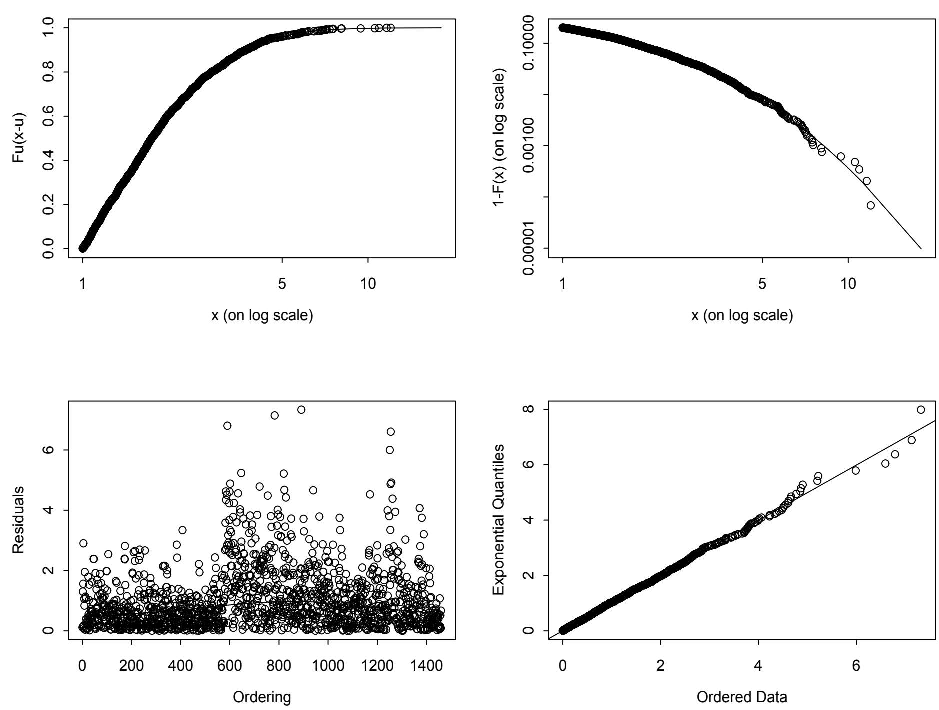
FTSE			
u	1.65		
ξ	0.15	0.06	2.49
β	0.82	0.06	12.81
NIKKEI			
u	2.28		
ξ	0.22	0.06	3.45
β	0.85	0.07	12.41

Note: The table reports the maximum likelihood estimation results of the GPD parameters which are used in constructing the semi-parametric estimation of the tail estimator. We estimate the GPD parameters using the 5% of the extreme observations.

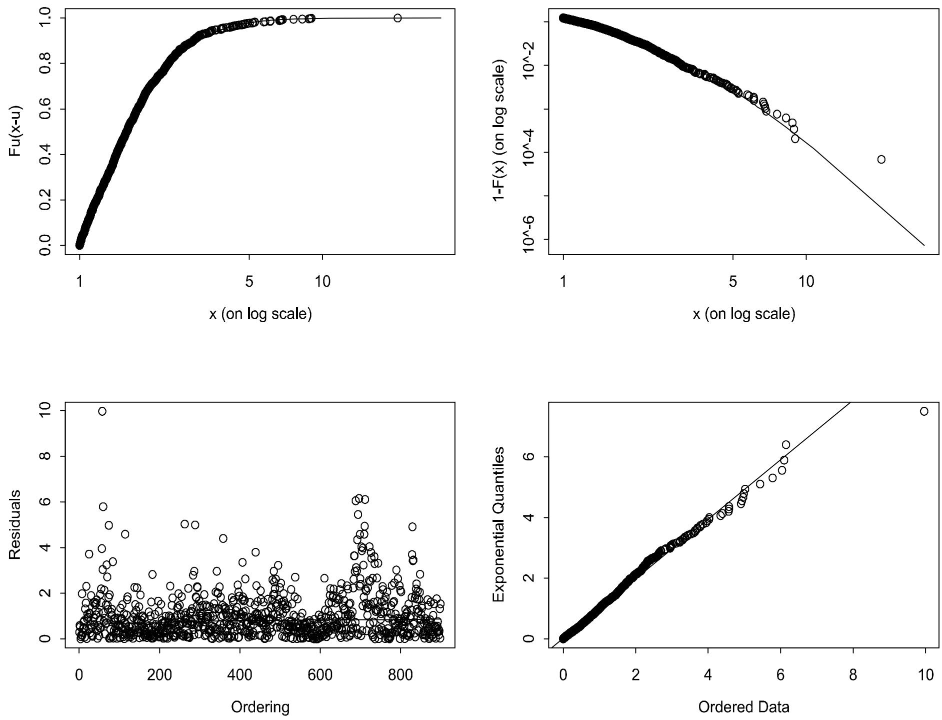
We report the graphical representations of the diagnostic tests for the GPD estimation results in Figure 3. The upper right panel graph for each market index is the representation of the GPD tail. The GPD specification approximates the excesses over the threshold remarkably well. The graphical representation of equation (3) is produced in the upper left panel of Figure 3.

[Figure 3] GPD estimation results

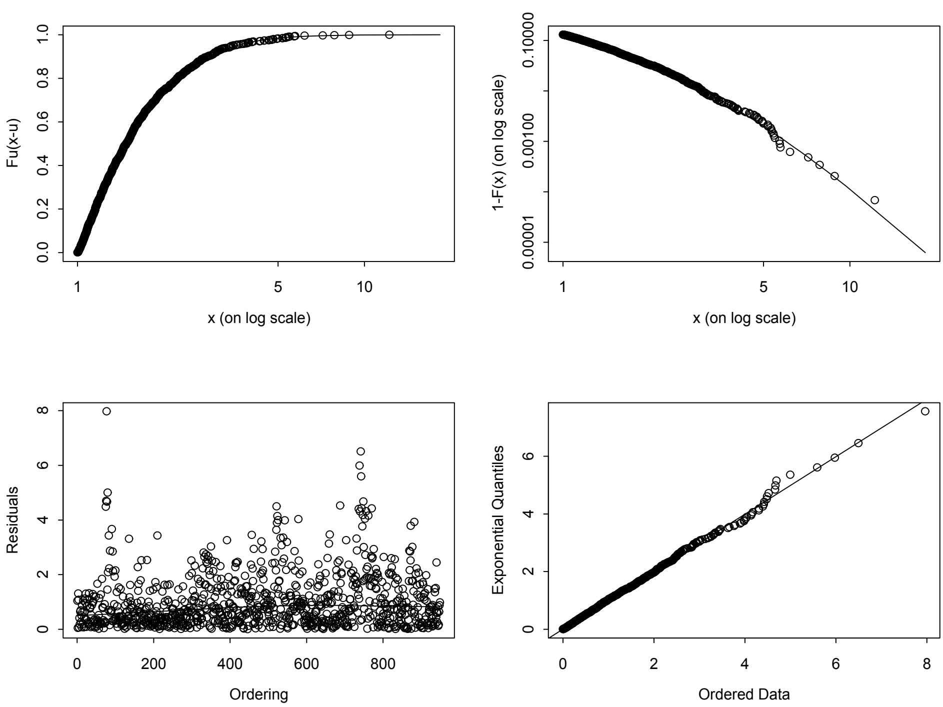
<KOSPI>



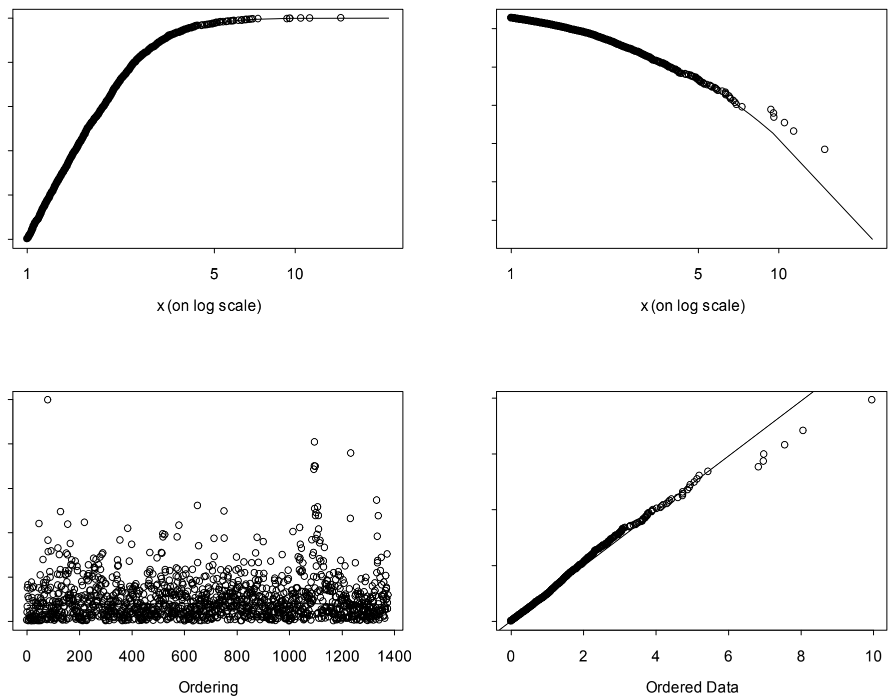
<S&P 500>


Fu(x-u)
x (on log scale)
1-F(x) (on log scale)
x (on log scale)
Residuals
Ordering
Exponential Quantiles
Ordered Data

<FTSE>


Fu(x-u)
x (on log scale)
1-F(x) (on log scale)
x (on log scale)
Residuals
Ordering
Exponential Quantiles
Ordered Data

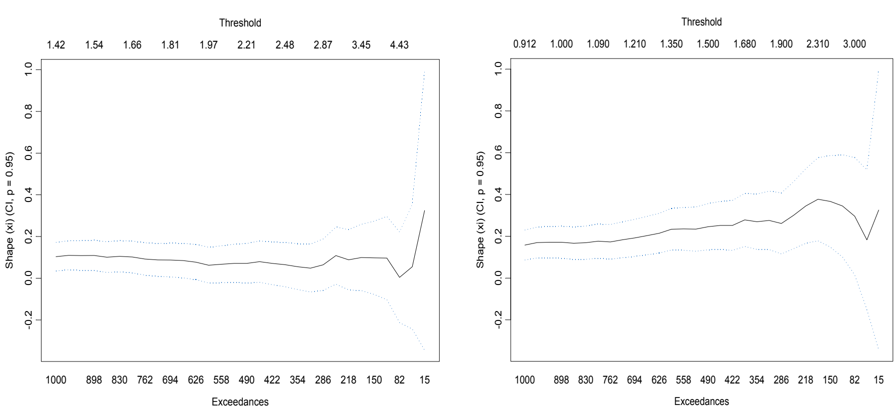
<NIKKEI>



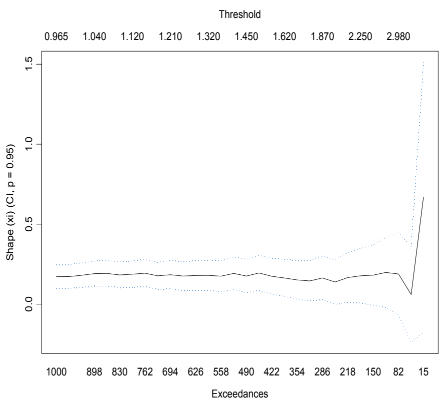
The upper left panel graph describes the excess distribution fit of the GPD. The GPD tail plot of the underlying distribution is presented in the upper right panel graph. In the lower left panel graph, we draw a scatterplot of the residuals. The lower right panel graph contains the qq-plot of the residuals.

[Figure 4] Estimates of the shape parameter ξ across a variety of threshold values u

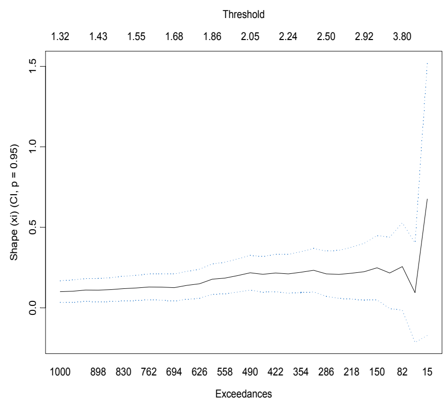
<KOSPI> <S&P 500>



<FTSE>



<NIKKEI>

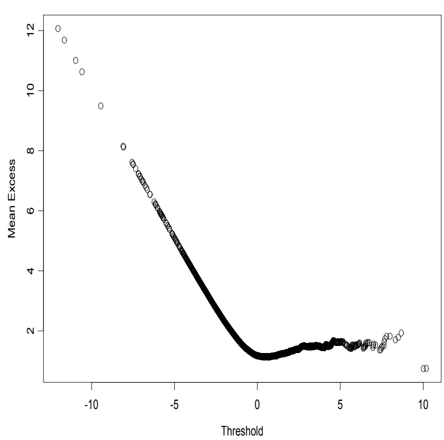


The shape parameter estimates remain positive across a variety of threshold values. The shape parameter estimates for the KOSPI and the FTSE returns are nearly constant, while the GPD estimates of tail behavior grow very slowly for the S&P500 and the NIKKEI returns. The estimates of ξ are robust to the choice of the threshold values as can be seen in Figure 4. As long as n_u is large enough to include 5% of the observations (about 367 observations), for example, the variance of the GPD estimator seems sufficiently low.

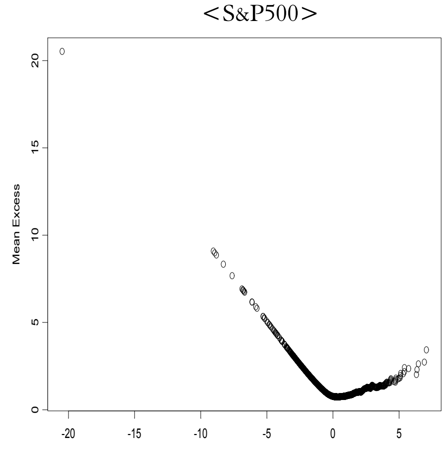
The GPD estimate of the tail shape parameter for each index return is shown in the figure. The estimated ξ's against the threshold values (upper-axis) are presented in solid line. The lower-axis corresponds to the number of exceedances of the excesses over the threshold. The dashed line represents the 95% confidence interval.

[Figure 5] Mean excess function plot of the threshold excesses

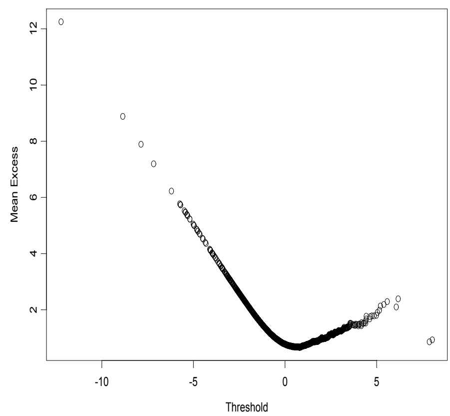
<KOSPI>



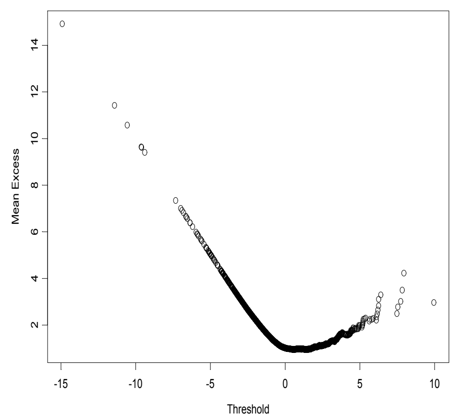
<S&P500>



<FTSE>



<NIKKEI>



We split the threshold excesses distribution into upper and lower tail and estimate the GPD parameters ξ and $\beta(u)$ of equation (3) by MLE. For the threshold excesses distribution, a graphcial inspection of the mean excess function plots in Figure 2 and Figure 5 sets the threshold values as approximately $u_{lower} = -1$ and $u_{upper} = 1$, respectively.

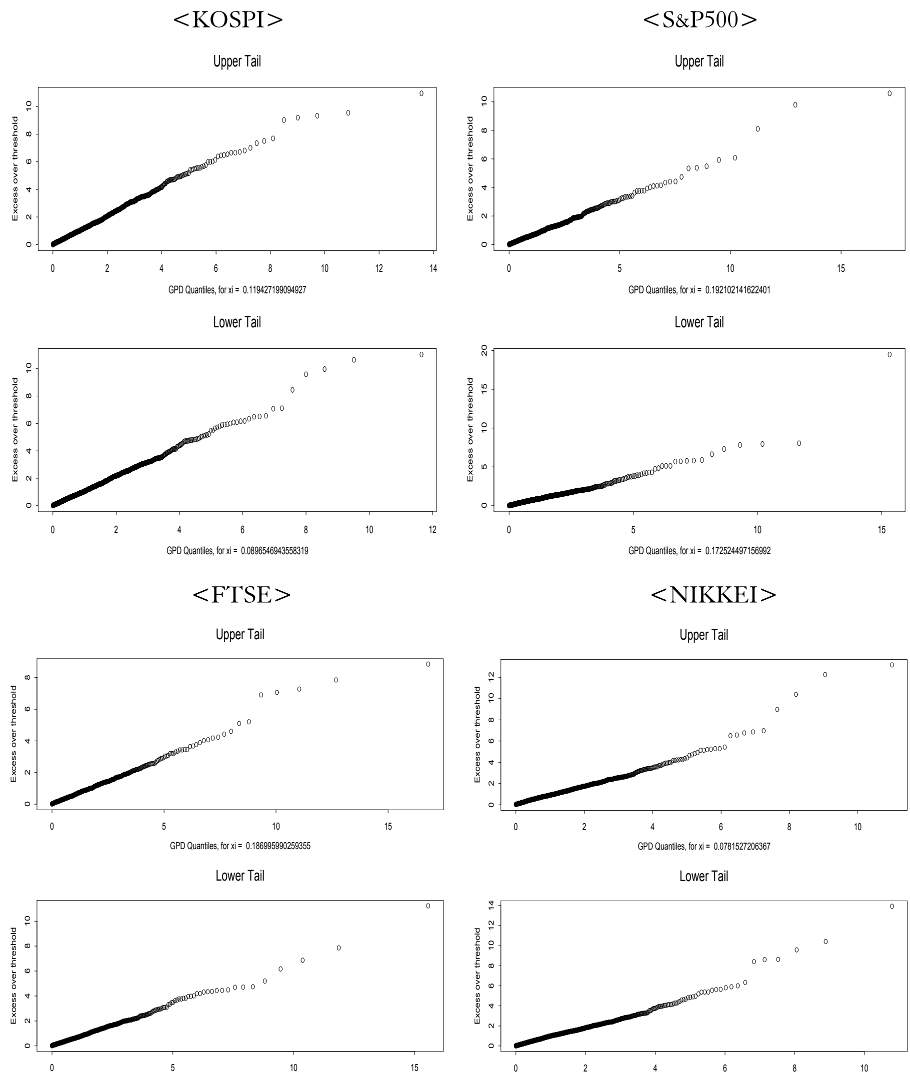
For the KOSPI with $u_{upper} = 1$, the mean excess function plot is linear with a positive slope, meaning that the threshold excesses are Pareto distributed in the upper tail. The tail shape parameters on both extreme deviations are calculated by the MLE methodology. With the upper threshold $u_{upper} = 1$, we include 20% of the total of 7,351 observations in the data and 22% of the data with the lower threshold $u_{lower} = -1$. The GPD parameter estimates in the upper tail are shown in Table 13. The qq-plots of the lower and upper tail shape parameter estimates in Figure 6 are close to linear when we use the GPD as the reference distribution. It seems reasonable to fit the tails of the threshold excesses on both extremes to the GPD function.

[Table 13] GPD $G_{\xi,\beta(u)}$ fit to the threshold excesses in the upper tail

	Value	Standard Error	t-ratio
KOSPI			
ξ	0.12	0.03	4.14
β	1.02	0.04	26.26
S&P 500			
ξ	0.19	0.04	5.15
β	0.63	0.03	20.60
FTSE			
ξ	0.17	0.04	4.57

β	0.65	0.03	20.16
NIKKEI			
ξ	0.07	0.03	2.93
β	0.94	0.03	27.02

[Figure 6] qq-plot of the upper and lower tail estimates of the GPD

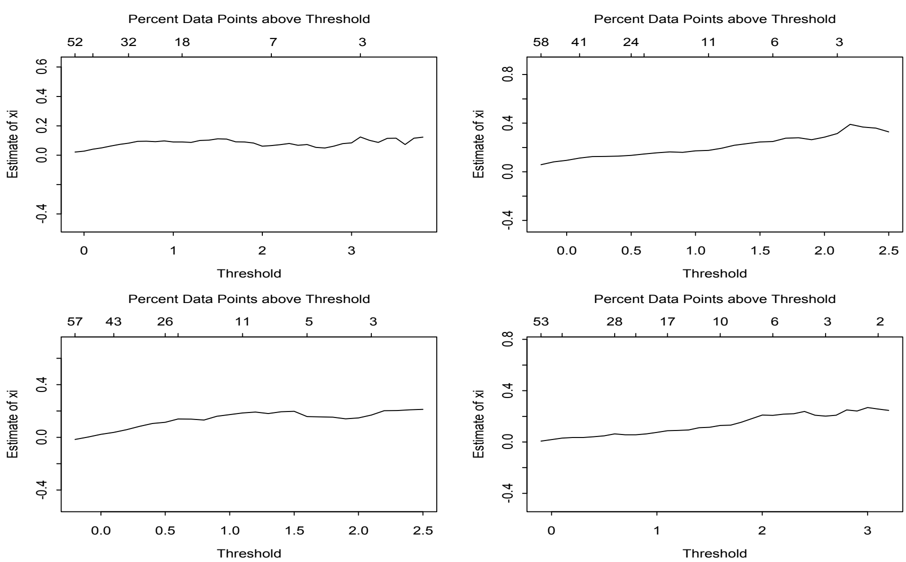


It is hard to pick up the level of a threshold where the mean excess function is linearly correlated with respect to the threshold for a fixed value of ξ. However, Hsu et al. (2012) find that including 5% of the observations is reasonable in estimating the GPD function and consistent with earlier studies. In that regard, we investigate the GPD specifications across a variety of threshold values. We set the initial value of the threshold as zero for the GPD model. This includes 48% of the total of 7,351 observations of the KOSPI data to fit the GPD, 46% of the S&P 500, 48% of the FTSE and 49% of the NIKKEI 225 data. We increase the threshold value consecutively to reach the point where the model estimation contains less than 1% of the total observations.

We find that the statistical level of significance has a tendency to move in the opposite direction with the level of threshold values. However, the values of the estimated parameters of the GPD are relatively constant across a variety of threshold values. The choice of an optimal threshold is an important issue as shown in Oh (2005), however, the GPD estimates from the data used in this paper are stable and robust to the choice of a variety of threshold values. This is in line with the result in McNeil and Frey (2000), where they show that the GPD estimator is efficient and stable with respect to the choice of the threshold value for the fat-tailed distributions in general.

As indicated in Figure 7, the estimates of ξ vary from 0 to 0.1 for the KOSPI, from 0 to 0.4 for the S&P 500, from 0 to 0.2 for the FTSE and from 0 to 0.25 for the NIKKEI 225. For comparison, we report the GPD estimation results in Table 14. The maximum likelihood estimates for ξ, $\beta(0)$ and the asymptotic standard errors with the threshold $u=0$ are presented in Table 14. The tail shape parameter estimates of ξ for each index returns range from 0.07 to 0.17. This can be interpreted as having a heavy tail for each index returns

[Figure 7] Estimates of the GPD parameter ξ across a variety of threshold values u



[Table 14] GPD $G_{\xi,\beta(u)}$ estimation results with a threshold value of $u=0$

	Value	Standard Error	t-ratio
KOSPI			
ξ	0.03	0.02	1.75
β	1.13	0.03	43.38
S&P 500			
ξ	0.09	0.02	5.66
β	0.72	0.02	41.86
FTSE			
ξ	0.02	0.01	1.64
β	0.78	0.02	45.43
NIKKEI			
ξ	0.02	0.01	1.29
β	1.02	0.02	45.30

Note: The table reports the maximum likelihood estimation results of the GPD parameters which are used in constructing the semi-parametric estimation of the tail estimator.

3.4. Risk Measure Evaluation Results

Table 15 reports the computation results of the VaR_α and the ES_α based on the GPD approximations (7) and (10), respectively.

[Table 15] Estimates of the VaR_α and ES_α with the GPD

	VaR_α	ES_α
KOSPI		
0.95	2.56	3.88
0.99	4.65	6.17
S&P 500		
0.95	1.71	2.74
0.99	3.29	4.66
FTSE		
0.95	1.67	2.60
0.99	3.10	4.33
NIKKEI		
0.95	2.30	3.43
0.99	4.08	5.34

According to the estimation results in Table 15, the daily loss on the S&P 500 index may be as low as -1.71% with the 5% probability. And, under the condition that the daily negative returns are less than -1.71%, the average loss on the S&P 500 index return is -2.74%. Also, the daily loss on the S&P 500 index may be as low as -3.29% with the 1% probability. Again, under the condition that the daily negative returns are less than -3.29%, the average loss on the index return is -4.66%. If the distribution of the daily losses is a fat-tailed distribution, the VaR_α and the ES_α computed from the standard normal distribution may be misleading and underestimate the true VaR_α and the ES_α. To confirm this assertion, we compute the VaR_α and the ES_α for $\alpha = 0.95,\ 0.99$. Table 16 shows the estimates of the VaR_α and ES_α based on the normal distribution. The daily negative return on the S&P 500 index could be as low as -1.87% with the 5% probability. And, under the condition that the daily negative returns are less than -1.87%, the average loss on the index return is -2.36%. Also, the daily loss on the index could be as low as -2.66% with 1% probability. And, under the condition that the daily negative returns are less than -2.66%, the average loss on the index return is -3.06%.

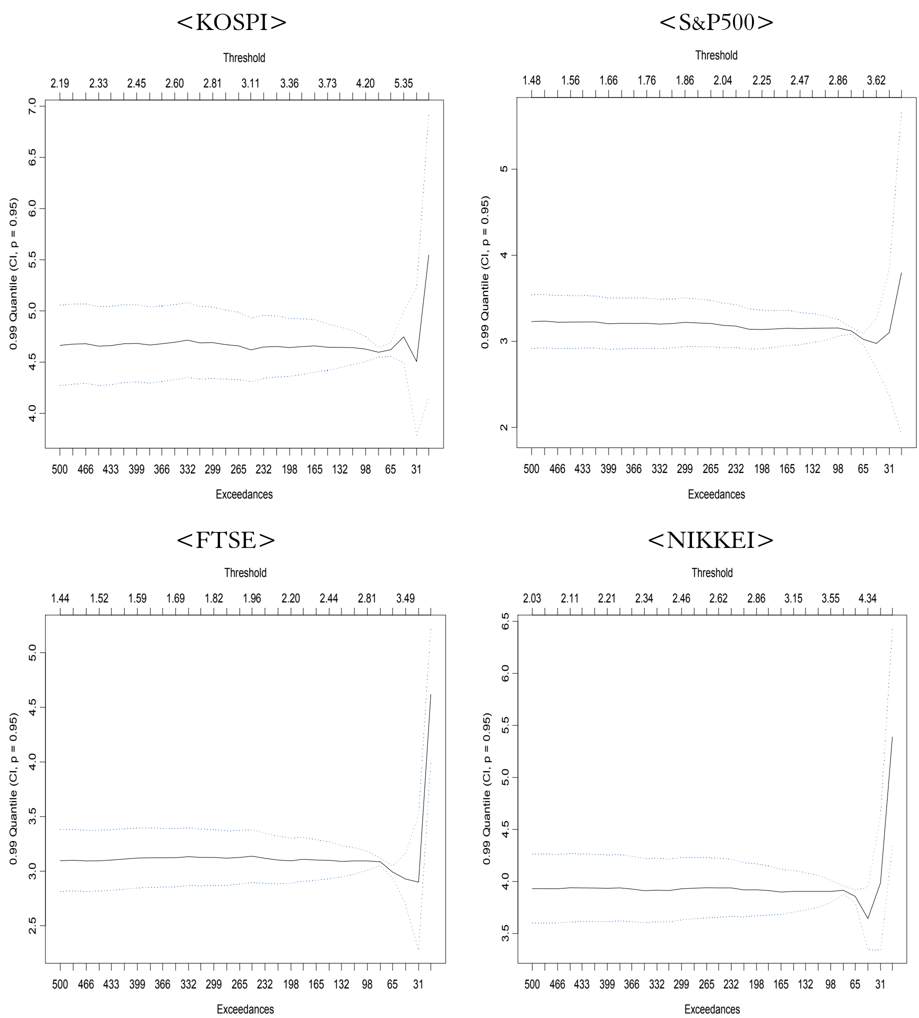
[Table 16] Estimates of the VaR_α and ES_α with the normal distribution

	VaR_α	ES_α
KOSPI		
0.95	2.71	3.41
0.99	3.86	4.42
S&P 500		
0.95	1.87	2.36
0.99	2.66	3.06
FTSE		
0.95	1.80	2.27
0.99	2.56	2.94
NIKKEI		
0.95	2.40	3.01
0.99	3.40	3.89

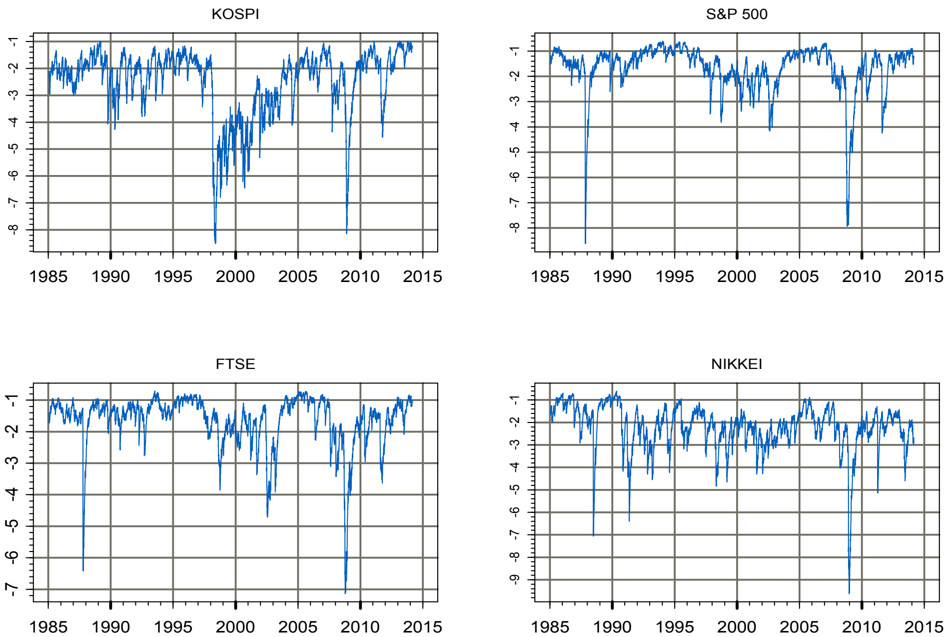
We compare the computation results of the VaR_α based on the GPD approximations to those based on the RiskMetrics methodology and the GARCH model estimation. We follow the procedure presented in Oh (2005) in calculating the tail-related risk measures. The robustness test results of the VaR_α estimates across a variety of threshold values u are presented in Figure 8. The estimates of the VaR_α are consistent for a variety of threshold values. Contrary to this, the

VaR values based on the RiskMetrics methodology and the GARCH model are extremely volatile as shown from Figure 9 to Figure 12. GARCH models are known to have poor tail properties and are not designed to evaluate the VaR. From a risk manager's perspective, it would be difficult to adjust the capital requirement of a financial institution to conditional market risk. The use of a conditional return distribution also raises concerns related to constructing conditional variance-covariance measures when a portfolio is composed of a large number of assets. Due to the concerns raised for practical and statistical reasons, we can conclude that the GPD method for measuring unconditional market risk is appropriate for measuring and managing the tail-related risk.

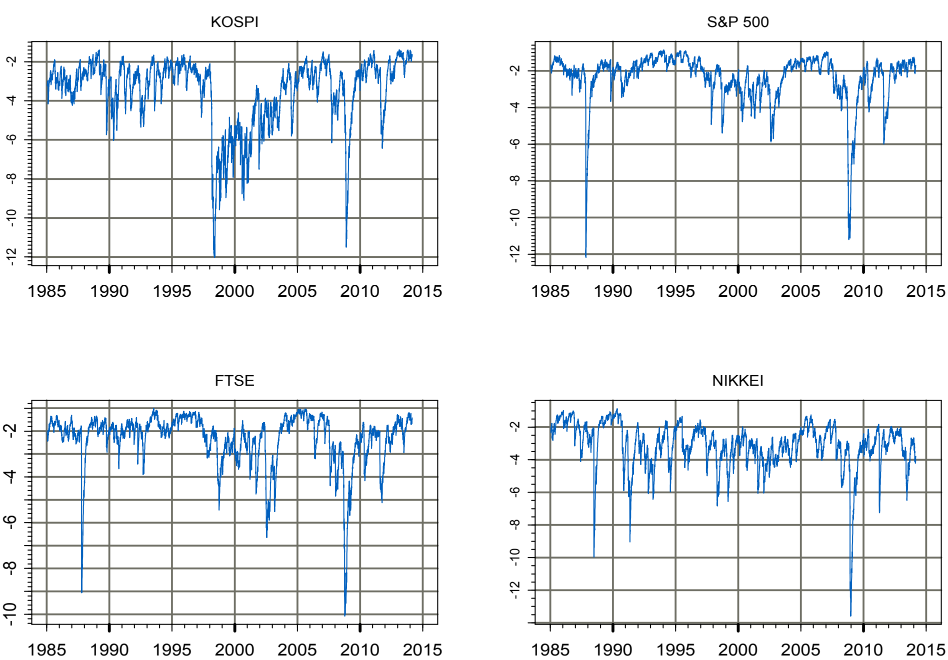
[Figure 8] Sensitivity analysis of the $VaR_{0.99}$ with respect to threshold values



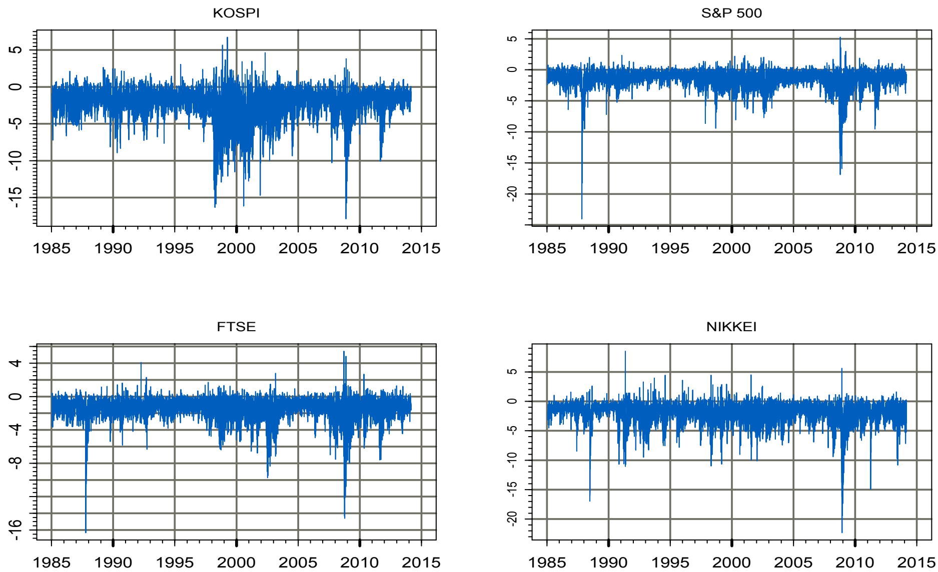
[Figure 9] $VaR_{0.95}$ based on RiskMetrics Method



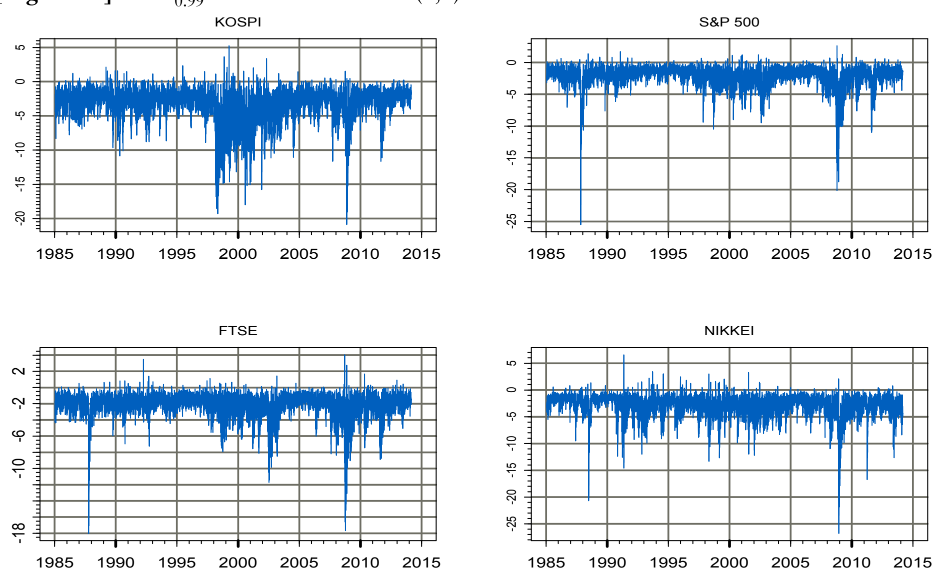
[Figure 10] $VaR_{0.99}$ based on RiskMetrics Method



[Figure 11] $VaR_{0.95}$ based on GARCH(1,1) model



[**Figure 12]** $VaR_{0.99}$ based on GARCH(1,1) model



IV. Concluding Remarks

In this paper, the EVT based semi-parametric approach to estimating the tail-related risk is compared with the method for evaluation of the extreme risk with normally distributed returns.

We fit the GEV distribution to the independent realizations of the block maxima random variable. The GEV distribution of various block maxima belongs to the Frechét distribution. We choose the number of observations per block to have the annual block maxima, the semester block maxima, the quarterly block maxima, the monthly block maxima and the weekly block maxima. The number of blocks for each block maxima is 30, 59, 117, 350 and 2,127, respectively. The tail shape parameter estimates of the market indices with a variety of block maxima are consistent for the KOSPI and the NIKKEI 225. For the S&P 500 and the FTSE, those range from 0.22 to 0.46 and have a tendency to increase with the number of observations in the block. For most indices, the tail shape parameter estimates with the weekly block maxima have negative values with large standard errors. In sum, we find that the tail shape parameter estimates of the GEV distribution with the block maxima are robust to the choice of the number of observations per block.

As an application of the GEV distribution parameter estimates, the out-of-sample forecast of the probability of a maximum loss at the 1-period horizon is made. Since the maximum of the monthly block maxima is 12.02% for the KOSPI, the out-of-sample forecast of the probability of a maximum loss at the 1-period horizon is calculated as 0.0054%. Armed with the parameter estimates of the GEV distribution, the 40-year return level is computed. The return level estimate of 15.53% for the KOSPI returns means that the monthly maximum loss observed during a period of one month will exceed 15.53% only in one month out of every 40 years, on average. From the point of view of a risk manager, it seems that the KOSPI is riskier than any other market indices.

As an alternative method for estimating the tail-related risk measures to the GEV distribution, we estimate the GPD function with the excesses over a high threshold. In theory, it is best to fit the GPD to the data solely pertained to the tail of the distribution and not included in the center of the distribution. However, it is hard to pick up the level of a threshold where the mean excess function is linearly correlated with respect to the threshold for a fixed value of the tail shape parameter. To balance between efficiency and unbiasedness, we need to find an optimal threshold over which we have enough observations to obtain efficient parameter estimates and minimize the bias of the GPD parameter estimates.

We investigate the GPD specifications across a variety of threshold values. We set the initial value of the threshold as zero for the GPD specification. This includes 48% of the total of 7,351 observations of the KOSPI data to fit the GPD, 46% of the S&P 500, 48% of the FTSE and 49% of the NIKKEI 225 data. We then increase the threshold value consecutively to reach the point where the model estimation contains less than 1% of the total observations. We find that the statistical level of significance has a tendency to move in the opposite direction with the level of threshold values. However, the values of the estimated parameters of the GPD are relatively constant across a variety of threshold values. The tail shape parameter estimates of the GPD vary from 0 to 0.1 for the KOSPI, from 0 to 0.4 for the S&P 500, from 0 to 0.2 for the FTSE and from 0 to 0.25 for the NIKKEI 225. Specifically, the tail shape parameter estimates with extreme 5% of the observations for each index returns range from 0.07 to 0.27. This suggests that the GPD with the index returns can be represented as having a heavy tail. The tail shape parameter estimates of the GPD are robust to the choice of the threshold values. As long as we choose a threshold high enough to include 5% of the observations (about 367 observations in our study), for example, the variance of the GPD estimator seems sufficiently low. The choice of an optimal threshold is an important issue as shown in Oh (2005), however, the GPD estimates from the data used in this paper are stable and robust to the choice of a variety of threshold values. This is in line with the result in McNeil and Frey (2000), where they show that the GPD estimator is efficient and stable with respect to the choice of the threshold value for the fat-tailed distributions in general.

Lastly, we estimate the tail-related risk measures such as the Value-at-Risk (VaR) and the expected shortfall (ES). Combined with the GPD estimator, we use the non-parametric method for estimating the risk measures using an empirical distribution. The GPD approach to estimating the *VaR* and the *ES* is compared with the method for evaluating the extreme risk with normally distributed returns. When the index returns have a fat-tailed distribution, the risk measures computed from the normal distribution underestimate the tail-related risk. The GPD estimator in this paper is proved to be efficient and stable with respect to the choice of the threshold value when the return has a fat-tailed distribution. We exercise the robustness test of the VaR_α estimates for a variety of threshold values. The estimates of the VaR_α are stable with respect to the changes in the threshold values. The GPD model is most appropriate in measuring and managing the risk associated with extreme events in terms of accuracy, stability and robustness. We compare the computation results of the VaR_α based on the GPD approximations to those based on the RiskMetrics methodology and the GARCH model estimation. The estimates of the VaR_α are robust to a variety of threshold values. Contrary to this, the *VaR* values based on the RiskMetrics methodology and the GARCH model are extremely volatile. From a risk manager's perspective, it would be difficult to adjust the capital requirement of a financial institution to conditional market risk. Due to the concerns raised for practical and statistical reasons, we can conclude that the GPD method for measuring unconditional market risk is appropriate for measuring and managing the tail-related risk.

References

Akgiray, V., C. G. Booth, and B. Seifert (1988), "Distribution Properties of Latin American Black Market Exchange Rates," *Journal of International Money and Finance*, 71 (1), 37-48.

Ang, A., and J. Chen (2002), "Asymmetric Correlations of Equity Portfolios," *Journal of Financial Economics*, 63 (3), 443-494.

Booth, G. G., J. P. Broussaard, T. Martikainen, and V. Pattonen (1997), "Prudent Margin Levels in the Finnish Stock Index Futures Market," *Management Science*, 43 (8), 1177-1188.

Carmona, R. (2004), Statistical Analysis of Financial Data in S-PLUS, Springer-Verlag, New York.

Carmona, R., and J. Morrisson (2001), Heavy Tails and Copulas with Evanesce, ORFE Tech, Report, Princeton University.

Coles, S. (2001), An Introduction to Statistical Modeling of Extreme Values, Spring-Verlag, London.

Dacorogna, M. M., U. A. Müller, O. V. Pictet, and C. G. de Vries (2001), "Extremal Forex Returns in Extremely Large Data Sets," *Extremes*, 4 (2), 105-127.

Danielsson, J., L. de Hann, L. Peng, and C. G. de Vries (2001), "Using a Bootstrap Method to Choose the Sample Fraction in Tail Index Estimation," *Journal of Multivariate Analysis*, 76, 226-248.

Danielsson, J., and C. G. de Vries (1997), Value-at-Risk and Extreme Returns, FMG-Discussion Paper No. 273, Financial Markets Group, London School of Economics.

________________ (1998), "Beyond the Sample: Extreme Quantile and Probability Estimation with Applications to Financial Data," Discussion paper, TI98-016/2. Tinbergen Institute.

Danielsson, J., B. N. Jorgensen, G. Samorodnitsky, M. Sarma, and C. G. de Vries (2013), "Fat Tails, VaR and Subadditivity," *Journal of Econometrics*, 172 (2), 283-291.

Diebold, F. X., T. Schuermann, and J. D. Stroughair (1997), "Pitfalls and Opportunities in the use of Extreme Value Theory in Risk Management," *Advances in Computational Management Science*, 2 (1), 3-12.

Embrechts, P., C. Klüppelberg, and T. Mikosch (1997), Modelling Extremal Events, Springer-Verlag, Berlin.

Embrechts, P., A. McNeil, and D. Straumann (2000), Correlation and Dependence in Risk Management: Properties and Pitfalls. In M. Dempster, H. K. Moffatt (Ed.), Risk Management: Value at Risk and Beyond, Cambridge University Press, Cambridge.

Fisher, R., and L. Tippett (1928), "Limiting forms of the Frequency Distribution of the Largest or Smallest Member of a Sample," *Proceedings of the Cambridge Philosophical Society*, 24 (2), 180-190.

Goldie, C., and R. L. Smith (1987), "Slow Variation with Reminder: Theory and Applications," *Quarterly Journal of Mathematics, Oxford 2nd Series* 38, 45-71.

Hall, P. (1990), "Using the Bootstrap to Estimate Mean Square Error and Select Smoothing Parameter in Nonparametric Problem," *Journal of Multivariate Analysis*, 32, 177-203.

Hols, M., and C. G. de Vries (1991), "The Limiting Distribution of Extremal Exchange Rate Returns," *Journal of Applied Econometrics*, 6 (3), 287-302.

Hsu, C-P., C.-W. Huang, and W. J. P. Chiou (2012), "Effectiveness of Copula-extreme Value Theory in Estimating Value-at-risk: Empirical Evidence from Asian Emerging Markets," *Review of Quantitative Finance and Accounting*, 39 (4), 447-468.

Jansen, D., and C. G. de Vries (1991), "On the Frequency of Large Stock Returns: Putting Booms and Busts into Perspective," *Review of Economics and Statistics*, 73 (1), 18-24.

Kearns, P., and A. Pagan (1997), "Estimating the Density Tail Index for Financial Time Series," *Review of Economics and Statistics*, 79 (2), 171-175.

Koedijk, K. G., M. M. A. Schafgans, and C. G. de Vries (1990), "The Tail Index of Exchange Rate Returns," *Journal of International Economics*, 29 (1), 93-108.

Loretan, M., and P. C. B. Phillips (1994), "Testing the Covariance Stationarity of Heavy-tailed Time Series," *Journal Empirical Finance*, 1 (2), 211-248.

Longin, F. M. (1996), "The Asymptotic Distribution of Extreme Stock Market Returns," *Journal of Business*, 63 (3), 383-408.

____________ (2000), "From Value-at-Risk to Stress Testing: The Extreme Value Approach," *Journal of Banking and Finance*, 24, 1097-1130.

Longin, F. M., and B. Solnik (2001), "Extreme Correlation of International Equity Markets," *Journal of Finance*, 56 (2), 649-676.

McNeil, A. J. (1998), Calculating Quantile Risk Measures for Financial Return Series using Extreme Value Theory. http://e-collection.ethbib.ethz.ch/show?type=bericht&nr=85.

____________ (1999), "Extreme Value Theory for Risk Managers," Internal Modelling and CAD II, RISK Books, 93-113.

McNeil, A. J., and R. Frey (2000), "Estimation of Tail-related Risk Measures for Heteroskedastic Financial Time Series: An Extreme Value Approach," *Journal of Empirical Finance*, 7, 271-300.

McNeil A. J., and T. Saladin (1997), "The Peaks over Thresholds Method for Estimating High Quantiles of Loss Distributions," Department of Mathematics, ETH Zentrum.

________________________ (2000), "Developing Scenarios for Future Extreme Losses using the POT Method," In Embrechts, P., (Ed.), Extremes and Integrated Risk Management, RISK Books, London.

Oh, S. (2005), "Extreme Value Theory and Value at Risk Focusing on GPD Models," *Journal of Money and Finance*, 19 (1), 72-114.

Tastan, H. (2006), "Estimating Time-varying Conditional Correlations between Stock and Foreign Exchange Markets," *Physica A: Statistical Mechanics and its Applications*, 360 (2), 445-458.

Zivot, E., and J. Wang (2006), "Modeling Financial Time Series with S-PLUS," Springer, 2006.