Assessing the Precision of Econometric Estimators by the Bootstrap Method

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I. Introduction

Asymptotic standard errors are used in practice to assess the precision or variability of various parameter estimates of a linear structural econometric model (SEM). Recently, "bootstrapping" has been put forward as an alternative method of evaluating the variability of an estimator; see Efron (1979, 1982), Freedman (1981), Efron and Gong (1983), and references therein.

The asymptotics of bootstrapping a linear SEM is discussed by Freedman (1983). He shows that the bootstrap gives a good approximation to the sampling distribution of two-stage least-squares (2SLS) estimates in stationary linear models. Some empirical results of bootstraping two and three-stage least squares (3SLS) estimates of a linear static model are presented in Peters (1981) and Freedman and Peters (1984a). They find the large-sample standard errors of 2SLS and 3SLS estimates of the Berndt-Wood (1975) model of the energy demand performing well. The model is static and the so-called restricted reduced form residuals (based on the 3SLS estimates of the model) are used in their bootstrapping.

In this paper we apply the bootstrap method to two-stage least squares estimates of a widely-known econometric model — Klein's (1950) Model I — and examine how large-sample standard errors of 2SLS estimates perform in a dynamic econometric model. We also consider two alternative sets of residuals in construction of the bootstrap data. One is the set of the restricted reduced form (RRF) residuals based on the structural parameter estimates. The other is the set of the least-squares residuals from the reduced-form equations, often referred to as the unrestricted reduced form (URF) residuals. Since the URF residuals are orthogonal to the

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predetermined variables while the RRF residuals are not, the former residuals preserve a key stochastic assumption of a SEM that exogenous variables are uncorrelated with the disturbance term.

The plan of the paper is as follows. In section 2 we outline the bootstrap in the context of estimating a linear dynamic SEM by 2SLS. Section 3 reports on bootstrapping Klein's Model I, and compares the results based on the URF and RRF residuals. Section 4 contains some concluding remarks.

2. Bootstrapping an Econometric Model

The idea of bootstrapping is to approximate the unknown distribution of the disturbance term by the empirical distribution of a set of residuals, construct the bootstrap data by repeated sampling from the latter distribution, and assess the variability in an estimate by Monte Carlo simulation. We describe the idea in the context of a linear dynamic SEM.

Consider a model of the form

$$\underset{\sim}{y}'_t \Gamma + \underset{\sim}{z}'_t A + \underset{\sim}{y}'_{t-1} C = \underset{\sim}{u}'_t, \quad t = 1, 2, \ldots T \tag{1}$$

where Γ, A, and C are matrices of unknown structural coefficients to be estimated, $\underset{\sim}{y}_t$ is a $G \times 1$ vector of current endogenous variables, $\underset{\sim}{z}_t$ is a $J \times 1$ vector of exogenous variables, and $\underset{\sim}{u}_t$ is a $G \times 1$ vector of structural disturbances, all at time t. A priori restrictions exist on Γ, A, and C so that all G structural equations are identified, and $|\Gamma| \neq 0$. The following assumptions are made about structural disturbances:

$$E\underset{\sim}{u}_t = \underset{\sim}{0} \qquad\qquad E\underset{\sim}{u}_t \underset{\sim}{u}'_s = \begin{cases} \Sigma & \text{for } t=s \\ 0 & \text{for } t \neq s \end{cases}$$

where Σ is a $G \times G$ contemporaneous covariance matrix.

The reduced form representation of the SEM in (1) is

$$\underset{\sim}{y}'_t = \underset{\sim}{z}'_t \Pi_1 + \underset{\sim}{y}'_{t-1} \Pi_2 + \underset{\sim}{v}'_t, \tag{2}$$

where $\Pi_1 = -A\Gamma^{-1}$ and $\Pi_2 = -C\Gamma^{-1}$ are matrices of reduced-form coefficients and $\underset{\sim}{v}'_t = \underset{\sim}{u}'_t \Gamma^{-1}$ is a $G \times 1$ vector of reduced-form disturbances. Clearly,

$$E\underset{\sim}{v}_t = \underset{\sim}{0}$$

and

$$E\underset{\sim}{v}_t \underset{\sim}{v}'_s = \begin{cases} (\Gamma^{-1})' \, \Sigma \, \Gamma^{-1} = \Omega & \text{for } t=s \\ 0 & \text{for } t \neq s. \end{cases}$$

Given the data on $(\underset{\sim}{y}_t', \underset{\sim}{z}_t')$ for $t = 1,2,\ldots, T$ and $\underset{\sim}{y}_0'$, the structural parameters in (1) can be estimated by 2SLS or other method. Structural disturbances may then be estimated by

$$\tilde{\underset{\sim}{u}}_t' = \underset{\sim}{y}_t' \tilde{\Gamma} + \underset{\sim}{z}_t' \tilde{A} + \underset{\sim}{y}_{t-1}' \tilde{C}, \qquad t=1, 2, \ldots\ldots, T \tag{3}$$

where $\tilde{\Gamma}$, $\tilde{A}$, and $\tilde{C}$ are the 2SLS estimates of Γ, A. and C, respectively. The estimates of the RF disturbances in (2) are obtained from

$$\begin{aligned} \tilde{\underset{\sim}{v}}_t' &= \underset{\sim}{y}_t' - \underset{\sim}{z}_t'\tilde{\Pi}_1 - \underset{\sim}{y}_{t-1}'\tilde{\Pi}_2 \\ &= \tilde{\underset{\sim}{u}}_t' \tilde{\Gamma}^{-1}, \qquad t=1, 2, \ldots, T \end{aligned} \tag{4}$$

where $\tilde{\Pi}_1 = -\tilde{A}\tilde{\Gamma}^{-1}$ and $\tilde{\Pi}_2 = -\tilde{C}\tilde{\Gamma}^{-1}$ provided that $|\tilde{\Gamma}| \neq 0$. These residuals are called the restricted reduced-form (RRF) residuals in the sense that they are obtained from the coefficient estimates satisfying the identification restrictions on the structural coefficients. Note that the RRF residuals are not necessarily orthogonal to exogenous variables although they have a zero mean if all structural equations contain intercept terms; see Park (1982). Freedman and Peters (1984a) have used the RRF residuals based on the 3SLS estimates of structural coefficients in bootstrapping the Berndt-Wood model.

Now we consider a model like (1) in which all parameters are known; the structural coefficients are equal to $\tilde{\Gamma}$, $\tilde{A}$, and $\tilde{C}$, and the reduced-form disturbances are independent and have a common distribution $\tilde{F}$. The common distribution $\tilde{F}$ is the empirical distribution of the RRF residuals with the probability mass of $1/T$ assigned to each of $\tilde{\underset{\sim}{v}}_t$, $t = 1, 2, \ldots, T$. Using this model, we obtain the bootstrap data by drawing the reduced-form disturbances $\tilde{\underset{\sim}{v}}_t^*$ from the distribution $\tilde{F}$ and generating the data on the endogenous variables recursively for $t = 1, 2, \ldots, T$, as

$$\underset{\sim}{y}_t^{*\prime} = \underset{\sim}{z}_t' \tilde{\Pi}_1 + \underset{\sim}{y}_{t-1}^{*\prime} \tilde{\Pi}_2 + \tilde{\underset{\sim}{v}}_t^{*\prime}. \tag{5}$$

The data on the exogenous variables are kept fixed, $\underset{\sim}{y}_0^* = \underset{\sim}{y}_0$, and the bootstrap data are denoted by asteriks.

Using the bootstrap data $\{(\underset{\sim}{y}_t^*, \underset{\sim}{z}_t), t = 1, \ldots, T\}$, we can obtain the 2SLS estimates $(\tilde{\Gamma}^*, \tilde{A}^*, \tilde{C}^*)$ of structural coefficients $(\tilde{\Gamma}, \tilde{B}, \tilde{C})$. The idea of bootstrapping is to take the distribution of $(\tilde{\Gamma}^* - \tilde{\Gamma}, \tilde{A}^* - \tilde{A}, \tilde{C}^* - \tilde{C})$ as an approximation to the distribution of $(\tilde{\Gamma} - \Gamma, \tilde{A} - A, \tilde{C} - C)$. Often the derivation of the former distribution is analytically intractable. Since $(\tilde{\Gamma}, \tilde{A}, \tilde{C})$ is in fact known, the distribution of $(\tilde{\Gamma}^* - \tilde{\Gamma}, \tilde{A}^* - \tilde{A}, \tilde{C}^* - \tilde{C})$ can be assessed by computer simulation of bootstrapping: we generate bootstrap samples many times by replicating random drawings from $\tilde{F}$ and infer about the sampling properties of the original 2SLS estimate by examining the empirical distribution of the bootstrap estimates.

An alternative method can be used for generating the bootstrap data. Let $\hat{\Pi}_1$ and $\hat{\Pi}_2$ be the least squares estimates of Π_1 and Π_2, respectively, of the reduced form equations in (2).
The least-squares residuals given by

$$\hat{\underset{\sim}{v}}'_t = \underset{\sim}{y}'_t - \underset{\sim}{z}'_t \hat{\Pi}_1 - \underset{\sim}{y}'_{t-1} \hat{\Pi}_2, \quad t=1, 2, \ldots, T \tag{6}$$

are often referred to as the unrestricted reduced-form (URF) residuals. They are unrestricted in the sense that the identifying restrictions on the structural parameters in (1) are not taken into account in obtaining $\hat{\Pi}_1$ and $\hat{\Pi}_2$. Unlike the RRF in (4), however, the URF residuals are orthogonal to the predetermined variables and have zero means when the reduced-form equations have intercept terms. It appears, that the URF residuals resemble the stochastic structure imposed on the disturbances more closely than the RRF.

The bootstrap data can be generated from the distribution $\hat{F}$ of the URF residuals with the mass of $1/T$ assigned to each of the URF residuals, $\hat{\underset{\sim}{v}}_t$, t = 1, 2, ..., T. Drawing a reduced-form disturbance vector $\hat{\underset{\sim}{v}}^*_t$ at random from $\hat{F}$ and using it in place of $\tilde{\underset{\sim}{v}}^*_t$ in (5), we construct the bootstrap data and obtain the 2SLS estimates of the "known" structural coefficients, $\tilde{\Gamma}$, $\tilde{A}$, and $\tilde{C}$.

3. Bootstrapping Klein's Model I

Klein's Model I as discussed in Theil (1971, pp. 432-458) is a dynamic simultaneous equation system consisting of the following six linear structural equations:

$$C_t = \alpha_0 + \alpha_1 P_t + \alpha_2 P_{t-1} + \alpha_3 (W_t + W'_t) + e_{t1}$$

$$I_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + \beta_3 K_{t-1} + e_{t2}$$

$$W_t = \gamma_0 + \gamma_1 X_t + \gamma_2 X_{t-1} + \gamma_3 (t-1931) + e_{t3}$$

$$X_t = C_t + T_t + G_t$$

$$P_t = X_t - W_t - T_t$$

$$K_t = K_{t-1} + I_t. \tag{7}$$

for t = 1921, 1922, ..., 1941. Endogenous variables are

C: consumption
P: profits
W: wage bill paid by the private sector
I: net investment
K: capital stock
X: total production of the private sector

and exogenous variables are

t: time
T: taxes
W': government wage bill
G: government nonwage bill
1: constant dummy

The annual data set for 1921-41 taken from Theil (1971, p. 456) yields the 2SLS estimates of the structural parameters and their large-sample standard errors as presented in columns (1) and (2) of Table 1.

Given the 2SLS estimates of the structural coefficients in eq. (7), we can compute the RRF residuals from the model and data. On the other hand, we can compute the URF residuals by regressing each of the six endogenous variables on all the predetermined variables of the model. Both URF and RRF residuals have zero means. The URF residuals, however, have smaller standard deviations than the RRF residuals because the former are the least squares residuals of reduced-form equations. The ratio of the standard deviaton of the RRF residuals to that of the URF residuals varies in Klein's data from 1.04 for the variable ranges I (net investment) to 1.20 for W (the private sector wage bill).

3.1. Bootstrapping with the URF Residuals

A bootstrap sample is constructed by obtaining pseudo — data on endogenous variables recursively using

$$\underset{\sim}{y}_t^{*\prime} = \underset{\sim}{z}_t' \hat{\Pi}_1 + \underset{\sim}{y}_{t-1}^{*\prime} \hat{\Pi}_2 + \hat{\underset{\sim}{v}}_t^{*\prime}, \quad t=1921, 1922, \ldots, 1941. \tag{8}$$

The coefficient matrices $\hat{\Pi}_1$ and $\hat{\Pi}_2$ are the least-squares estimates of the reduced-form coefficients in eq. (2) and $\hat{\underset{\sim}{v}}_t^*$ is a random vector drawn from the empirical distribution $\hat{F}$ of the URF residuals. The data on exogenous variables are kept fixed. The 2SLS estimates of "known" structural coefficients — as given in the first column of Table 1 — and their standard errors

[Table 1]

2SLS Estimates of Klein's Model I and the Bootstrap Results Using URF Residuals

		2SLS		Bootstrap (URF Residuals)				
Equation	Coefficient	(1) Estimates	(2) Standard Error	(3) Mean	(4) Stand. Dev.	(5) t-ratio	(6) RMS	(7) (6)/(4)
Consumption	α_0	16.5547	1.3208	16.0194	1.7602	−6.08	1.7884	1.02
C	α_1	.0173	.1180	.0605	.1453	5.95	.1326	.91
	α_2	.2162	.1072	.2049	.0945	−2.39	.1168	1.24
	α_3	.8102	.0402	.8108	.0322	.37	.0435	1.35
Investment	β_0	20.2782	7.5428	17.8927	6.2539	−7.63	6.8845	1.10
I	β_1	.1502	.1732	.3316	.1390	26.11	.1473	1.06
	β_2	.6159	.1628	.4431	.1157	−29.86	.1319	1.14
	β_3	−.1578	.0361	−.1470	.0281	7.68	.0311	1.11
Labour Demand	γ_0	1.5003	1.1472	2.3398	1.2226	13.73	1.6777	1.37
W	γ_1	.4389	.0363	.4578	.0427	8.86	.0412	.97
	γ_2	.1467	.0387	.1125	.0363	−18.82	.0429	1.18
	γ_3	.1304	.0292	.1399	.0235	8.08	.0370	1.57

Note: Bootstrap results are based on 400 replications.

can then be obtained from the bootstrap sample.

This procedure may be repeated a number of times. On each replication a new set of residual vectors are drawn, a new sample is constructed, and the 2SLS estimates and their asymptotic standard errors are computed. Columns (3) and (4) of Table 1 present, for each structural coefficient of the model in column (1), the mean and standard deviation of 2SLS estimates computed from 400 bootstrap samples. For example, for the parameter α_1 with the original 2SLS estimate of 0.0173, the bootstrap has yielded the mean and standard deviation of 0.0605 and 0.1453, respectively.

The difference between the mean and the original 2SLS estimate in this example of α_1 seems to be appreciable. A statistically significant difference between the two would indicate a small-sample bias in the 2SLS estimate. If we denote the 2SLS estimate of a coefficient in column (1) by $\hat{\theta}$, and its corresponding mean and standard deviation in columns (3) and (4) by $\bar{\theta}^*$ and s^*_θ, respectively, then the ratio

$$t = T^{1/2} (\bar{\theta}^* - \hat{\theta})/s^*_\theta ,$$

where T = 400, may be computed to check for the bias in the 2SLS estimate. Thus, for α_1 we obtain the t value of 20 (.0605 − .0173)/.1453 = 5.95 on 399 degrees of freedom. The t value indicates at the 0.01 level of significance that the 2SLS estimate is significantly biased. [Unless otherwise noted, the level of significance used in this paper is 0.01.] The t ratios are presented in the fifth column of Table 1. The 2SLS estimate has a statistically significant bias in almost all cases.

The sample standard deviation in the fourth column of Table 1 gives the bootstrap estimate of the variability or precision of the 2SLS estimate in column (1) and provides an alternative to the large-sample standard error in column (2).

For example, the standard deviation of .1453 measures the variability of the 2SLS estimates of α_1 in the simulation and is a bootstrap estimate of the standard deviation of the original 2SLS estimate .0173. On the other hand, the conventional asymptotics yields the standard error of .1180, and suggest less variability of the 2SLS estimate than the bootstrap.
However, the large-sample standard error in column (2) is smaller than the standard deviation in column (4) only in 4 out of 12 cases, and the ratio of the former to the latter ranges from .75 for α_0 to 1.28 for β_2.

A measure of the variability of large-sample standard errors is also provided by the bootstrap samples. Let s_i stand for the large-sample standard error of the 2SLS estimate of a given structural coefficient computed from the i-th bootstrap sample data. The variability of the large-sample standard errors can be measured by the root mean squares (RMS):

$$\mathrm{RMS} = (\Sigma\ s_i^2/400)^{1/2}.$$

For example, RMS is .1326 for α_1 as presented in column (6). This may be interpreted as the large-sample standard error in a typical sample that may be compared to the standard deviation of .1453 in the fourth column, and suggests that the large-sample standard error underestimates the real variability by a factor of about 0.09.

Comparing columns (4) and (6) in Table 1, we find that the large-sample standard error tends to overstate the variability of the 2SLS estimates in many cases. Column (7) gives the ratio of the RMS in column (6) to the standard deviation in column (4). The ratio ranges in value from 0.91 for α_1 to 1.57 for α_3, and the ratio is greater than one in all but two cases. The conventional asymptotic standard error seems to provide a reasonably good measure of the variability in 2SLS estimates.

3.2. Bootstrapping with the RRF Residuals

We now turn to bootstrapping with the RRF residuals. Table 2 presents for each structural coefficient of the model the mean and standard deviation of coefficient estimates computed from 400 bootstrap sample using the RRF residuals. The mean in column (3) reveals the bias in the 2SLS estimates while the standard deviation in column (4) measures the variability of the 2SLS estimates.

The t ratio in the fifth column indicates that the mean in column (3) is significantly different from the parameter value in column (1) in 5 out of 12 cases. It is seen, therefore, that whether the URF or RRF residuals are used in bootstrapping ,the mean of the bootstrap coefficient estimates can be substantially different from the original 2SLS coefficient. Such a strong bias in 2SLS estimate may be due to the presence of lagged endogenous variables in the structural equations.

The standard deviation in column (4) of Table 2 is smaller than the large-sample standard error in column (2)in 8 out of12 cases.The ratio of the one to the latter ranges from .63 for β_1 to 1.25 for α_0, indicating that the asymptotic standard error may not be too overly optimistic or cautious in assessing the precision of the 2SLS estimate. The RMS of large-sample standard errors reported in column (6) of Table 2 is seen to be very close to the standard deviation in column (4) in almost all cases. The ratio of the entries in columns (6) to (4) as presented in column (7) ranges from 0.89 for α_0 to 1.15 for β_1. The conventional large-sample standard error appears to be doing well. This is in agreement with the result obtained by Freedman and Peters (1984a) for 2SLS. They used, however, the RRF

[Table 2]

2SLS Estimates of Klein's Model I and the Bootstrap Results Using RRF Residuals

		2SLS		Bootstrap (RRF Residuals)				
Equation	Coefficient	(1) Estimates	(2) Standard Error	(3) Mean	(4) Stand. Dev.	(5) t-ratio	(6) RMS	(7) (6)/(4)
Consumption	α_0	16.5547	1.3208	16.6500	1.6470	1.16	1.4661	.89
C	α_1	.0173	.1180	.0981	.0868	18.62	.0965	1.11
	α_2	.2162	.1072	.1072	.0857	−17.64	.0873	1.02
	α_3	.8102	.0402	.8059	.0414	−2.08	.0369	.89
Investment	β_0	20.2782	7.5428	19.6731	6.9509	−1.74	7.1469	1.03
I	β_1	.1502	.1732	.2528	.1096	18.72	.1263	1.15
	β_2	.6159	.1628	.5276	.1083	−16.30	.1150	1.06
	β_3	−.1578	.0361	−.1561	.0328	1.04	.0339	1.03
Labour Demand	γ_0	1.5003	1.1472	1.3672	1.3559	−1.96	1.4261	1.05
W	γ_1	.4389	.0363	.4446	.0299	3.81	.0319	1.06
	γ_2	.1467	.0387	.1427	.0340	−2.35	.0354	.94
	γ_3	.1304	.0292	.1282	.0310	−1.42	.0308	.99

Note: Bootstrap results are based on 400 replications.

residuals based on the 3SLS estimates of structural coefficients in constructing bootstrap samples.

4. Concluding Remarks

In this paper the variability or precision of the asymptotic standard error of the 2SLS estimates has been examined by bootstrapping a widely known dynamic SEM. Although the bootstrap results could be different depending on whether URF or RRF residuals are used, the differences are not found to be of practical significance.

The main finding from the bootstrap experiments is that the conventional asymptotic standard error performs well in assessing the real variability of the 2SLS estimates. One problem, both for the asymptotic formula and for the bootstrap, is that the residuals may tend to be smaller than the true disturbances due to the effect of fitting. Some inflation of the residuals may be appropriate to compensate for the deflation of the residuals. There is no generally valid rule. More experience with bootstrapping is desirable.

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