

A Theoretical Analysis of the Efficient Provision of a Public Good

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I. Introduction

The free-rider problem has long been regarded as intrinsically unsolvable as a practical manner. The essence of this problem is that for a Pareto-optimal solution to be reached, individuals must reveal their preferences for public goods. However since each individual consumes the total quantity of public goods supplied, it is in any individual's interest to either understate or overstate his preferences depending on how the stated preferences are used in the financing process of the public good. We are thus left without a direct way to determine the efficient level at which to produce a public good.

A number of recent papers have developed a variety of ways to overcome the free-rider problem. However the literature on the theoretical design of the mechanisms has not dealt with empirical issues, nor used real world data. The purpose of this paper is to analyze the efficient provision of a public good using the weak complementarity approach of Bradford and Hildebrandt (1977) and Hildebrandt and Tregarthen (1979-hereafter H-T).

This paper is organized as follows. In section II, it is shown that different individual behaviors (altruistic as opposed to self-interested behavior) toward financing a public good result in different amounts of willingnesses-to-pay for a public good. It is also shown that constant-income-elasticity private good demand functions satisfy the restriction that the marginal rate of substitution of private good price for the level of public good provision is independent of income. This fact is applied to the case of substitutes, which raises a strong possibility that this approach may be extended to a wide range of goods. Section III describes further directions of research.

II. Theoretical Issues

H-T estimated the efficient levels of school quality and other local public goods with data first used by Oates (1969). Based on the theoretical treatment of the weak complementarity approach in Bradhord and Hildebrandt (1977), they estimated a demand function for housing and with that calculated the efficient levels of public goods provision. They concluded that a method of

estimating \bar{p} , the price at which a public good becomes valueless, would greatly increase the power of the analysis. In this section it is shown first that it is not always necessary to calculate \bar{p} to get the efficient level of a public good.

Let's assume first that consumption of a private good and a public good are interdependent so there is a price of the private good, \bar{p} , at which the public good is valueless:

$$V_q^i(p, q, y^i) / V_y^i(p, q, y^i) = 0 \quad \text{at } p = \bar{p}, \quad (1)$$

where $V^i(p, q, y^i)$ is individual i 's indirect utility function. The second assumed restriction is that the marginal rate of substitution of the private good price for the level of the public good provided is independent of income:

$$\partial [V_q^i(p, q, y^i) / V_p^i(p, q, y^i)] / \partial y^i = 0, \quad i = 1, 2, \dots, n. \quad (2)$$

From the private good demand function,

$$\begin{aligned} \int_0^{x^i} p(\xi^i, q, y^i) d\xi^i &= \int_p^\infty x^i(\tau, q, y^i) d\tau + x^i p \\ \Rightarrow \int_0^{x^i} p(\xi^i, q, y^i) d\xi^i &= \int_p^{\bar{p}} x^i(\tau, q, y^i) d\tau \\ &\quad + \int_{\bar{p}}^\infty x^i(\tau, q, y^i) d\tau + x^i p \\ \Rightarrow \int_0^{x^i} p_q(\xi^i, q, y^i) d\xi^i &= \int_p^{\bar{p}} x_q^i(\tau, q, y^i) d\tau \\ &\quad + \int_{\bar{p}}^\infty x_q^i(\tau, q, y^i) d\tau. \end{aligned}$$

Proposition 1: If the two restrictions mentioned above are satisfied, then

$$\begin{aligned} \int_p^\infty x_q^i(\tau, q, y^i) d\tau &= 0 \quad \text{and} \quad \int_0^{x^i} p_q(\xi^i, q, y^i) d\xi^i \\ &= \int_p^{\bar{p}} x_q^i(\tau, q, y^i) d\tau. \end{aligned}$$

Proof:

$$x^i = -V_p^i / V_y^i \quad (\text{by Roy's identity}), \quad (3)$$

$$x_q^i = -\partial(V_p^i / V_y^i) / \partial q = -(V_{pq}^i V_y^i - V_p^i V_{qy}^i) / (V_y^i)^2. \quad (4)$$

By the restriction of interdependence,

$$V_q^i / V_y^i = 0 \quad \text{at } p = \bar{p}. \quad \text{Then } \partial(V_q^i / V_y^i) / \partial p = 0 \quad \text{at } p = \bar{p}. \quad (5)$$

From (5),

$$(V_{pq}^i V_y^i - V_q^i V_{py}^i) / (V_y^i)^2 = 0 \quad \text{at} \quad p = \bar{p}.$$

From the second restriction

$$\begin{aligned} \partial (V_q^i / V_p^i) / \partial y^i = 0 &\leftrightarrow (V_{qy}^i V_p^i - V_q^i V_{py}^i) / (V_p^i)^2 = 0 \\ &\leftrightarrow V_{qy}^i V_p^i = V_q^i V_{py}^i. \end{aligned}$$

From (4), (6) and (7),

$$x_q^i = 0 \quad \text{at} \quad p = \bar{p}.$$

Therefore,

$$\int_{\bar{p}}^{\infty} x_q^i = 0 \quad \text{and} \quad \int_0^{x^i} p_q(\xi^i, q, y^i) d\xi^i = \int_{\bar{p}}^{\bar{p}} x_q^i(\tau, q, y^i) d\tau.$$

This proposition shows that we can use either expression for the individual i 's willingness-to-pay.

Remark: If the assumption of constant marginal utility of income is adopted, the proof is much simpler:

$$\begin{aligned} V_q^i(p, q, y^i) = 0 \quad \text{at} \quad p = \bar{p} \quad \text{and} \quad V_{py}^i = V_{qy}^i = 0, \quad \text{then} \\ \partial V_q^i / \partial p = 0 \quad \text{at} \quad p = \bar{p}, \quad \partial x^i / \partial q = 0 \quad \text{at} \quad p = \bar{p}. \end{aligned}$$

In our model, two restrictions are imposed. First, the marginal rate of substitution of private good price for the level of public good provision is independent of income (same as before). The significance of this restriction is that even though there are income effects there will be no difference between the willingness-to-pay and the money equivalent of a utility change, both of which are represented by the change in the areas under the demand curve [Bradford and Hildebrandt (1977), p. 123]. Second, there exists \bar{p} at which the valuation of a public good becomes invariant and the partial derivative of the value of the public good with respect to private good price will be zero at \bar{p} . This restriction allows the possibility that at \bar{p} one may still place some positive value on the public good, which is a generalized version of the interdependency assumption made by H-T in the following sense. Their first restriction, (1), is based essentially on the self-interested individuals' behavior. Even though an individual has positive demand for a private good, a public good becomes

valueless at a specific price level. However since individuals' altruistic behavior (or public regardedness behavior¹⁾) may have an important role in the provision of a public good, it is likely that an individual will still place

- 1) According to Hanushek (1975), some individuals are inclined to support public expenditures above those indicated by their own private interests and this public regardedness is linked to their heritage and to their ethnic background. Also according to Deacon (1977), the appearance of certain anomalous results in the study (e.g., high incomes voter favoring welfare proposals), seemingly at odds with narrow self-interest, led to the introduction of "public regardingness" as an attribute that influences the behavior of certain classes of voters.
- 2) It can be shown that the usual willingness-to-pay expression in terms of the direct utility function ($= U_x/U_y$) is equivalent to the indirect utility expression ($= V_x/V_y$).

Let's consider the following constrained maximization problem:

$$\begin{aligned} \text{Max } U(x, q, N) \\ \text{s. t. } px + H = y, \end{aligned}$$

where N is a numeraire goods. Let

$$U(x, q, N) \equiv \bar{U}. \quad (2-1)$$

Total differentiation of (2-1) gives

$$U_x dx + U_q dq + U_N dN = 0. \quad (2-2)$$

From the budget constraint (with p fixed),

$$dN = dy - p dx. \quad (2-3)$$

Marginal utility of the numeraire good is equivalent to marginal utility of income, i.e.,

$$U_N = U_y. \quad (2-4)$$

From (2-2), (2-3) and (2-4), we get

$$\begin{aligned} U_x dx + U_q dq + U_y (dy - p dx) &= 0 \\ (U_x - p U_y) dx + U_q dq + U_y dy &= 0 \end{aligned} \quad (2-5)$$

Since efficiency requires that U_x/U_y be equal to p , the first term of expression (2-5) vanishes:

$$-\frac{dy}{dq} \Big|_{U=\bar{U}} = \frac{U_q}{U_y} \quad (2-6)$$

From the indirect utility function, let

$$V(p, q, y) \equiv \bar{U} \quad (2-7)$$

Total differentiation of (2-7) gives

$$V_p dp + V_q dq + V_y dy = 0 \quad (2-8)$$

Since $dp = 0$ (i.e., p is fixed), we have the following relation:

$$-\frac{dy}{dq} \Big|_{V=\bar{U}} = \frac{V_q}{V_y}. \quad (2-9)$$

Therefore from expressions (2-6) and (2-9),

$$-\frac{dy}{dq} \Big|_{U=V=\bar{U}} = \frac{U_q}{U_y} = \frac{V_q}{V_y}.$$

some valuation on the public good at a threshold price of the private good. Therefore it is assumed here that an individual's marginal valuation of the public good stays constant with respect to the private good price at the threshold price level. This assumption satisfies assumption (1) as a special case. Specifically, these two restrictions are expressed as follows:

$$\partial (V_q^i / V_p^i) / \partial y^i = 0^2, \quad (8)$$

$$\partial (V_q^i / V_y^i) / \partial p = 0 \quad \text{at } p = \bar{p} \quad (9)$$

where \bar{p} is the threshold price for the private good. If the two expressions (8) and (9) are satisfied, then x_y will be equal to zero at $p = \bar{p}$ from equations (4), (6), and (7). This implies that even though there is some value from the public good at \bar{p} Marshallian demand for x will not change with the change in the level of the public good.

Altruistic and self-interested behavioral assumptions generally yield different threshold prices. Let's consider the following indirect utility function which satisfies assumption (2):

$$V(p, q, y) = f(y) + g(p, q).$$

With assumption (1),

$$\frac{V_q(p, q, y)}{V_y(p, q, y)} = 0 \quad \text{at } p = \bar{p} \leftrightarrow \frac{g_p(p, q)}{f_y(y)} = 0 \quad \text{at } p = \bar{p} \quad (10)$$

With assumption (9),

$$\frac{g_{qp}(p, q) f_y(y) - g_q(p, q) f_{yp}(y)}{\{f_y(y)\}^2} = 0 \quad \text{at } p = \bar{p} \quad (11)$$

$$\leftrightarrow g_{qp}(p, q) = 0 \quad \text{at } p = \bar{p}$$

Therefore \bar{p} is generally different from \bar{p} , and the numerical values of \bar{p} and \bar{p} depend on the functional form of $g(p, q)$. Now the relationship between the threshold prices (\bar{p} and \bar{p}) and the change in the private good demand due to change in the public good is as follows:

$$\frac{V_q(p, q, y)}{V_y(p, q, y)} - \frac{V_q(p', q, y)}{V_y(p', q, y)} = \int_{\bar{p}}^{\bar{p}'} - \frac{\partial(V_q / V_y)}{\partial \tau} d\tau \quad (12)$$

Since $\partial (V_q / V_y) / \partial y$ is equal to zero from assumption (2), V_{qy} / V_p is equal to V_q / V_{py} . And

$$-\frac{\partial (V_q/V_y)}{\partial p} = -\frac{(V_{qp} V_y - V_q V_{yp})}{V_y^2},$$

$$\frac{\partial x}{\partial q} = -\frac{\partial (V_p/V_y)}{\partial q} = -\frac{(V_{pq} V_y - V_p V_{yq})}{V_y^2}$$

Therefore, $-\partial (V_q/V_y)/\partial p = \partial x/\partial q$ from assumption (2):

$$\int_p^{p'} -\frac{\partial (V_q/V_y)}{\partial \tau} d\tau = \int_p^{p'} x_q(\tau, q, y) d\tau.$$

This implies that $\partial (V_q/V_y)/\partial p$ is equal to zero at $p=p'$ if and only if x_q is equal to zero at $p=p'$. With assumption (9), $\partial x/\partial q$ is equal to zero at \bar{p} . On the other hand if assumption (1) holds, then assumption (9) always holds at \bar{p} , but not vice versa.

It is also noteworthy to compare the two assumptions, (1) and (9), based on the properties of an indirect utility function. Let's define complements first. If a private good demand (x) increases as a public good (q) increases, i.e., $\partial x/\partial q > 0$, then they are defined as complements in consumption.³⁾

From the indirect utility function (10),

$$x_q = -\frac{\partial (V_p/V_y)}{\partial q} = -\frac{(V_{pq} V_y - V_p V_{yq})}{V_y^2},$$

$$V_{yq} = 0 \quad X_q = -\frac{V_{pq}}{V_y}.$$

In the case of complements, $x_q > 0$ (or equivalently $V_{pq} < 0$). Let's assume $V_q > 0$ for all positive p . Then

$$X_{qp} = -\frac{\partial (V_{pq}/V_y)}{\partial p} = -\frac{(V_{ppq} V_y - V_{pq} V_{yp})}{V_y^2} = -\frac{V_{ppq}}{V_y}.$$

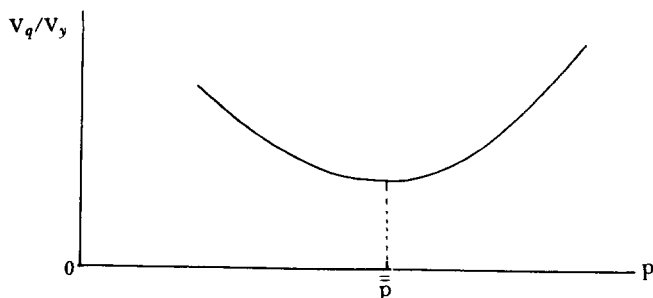


Figure 1

3) This definition is different from one in Deaton and Muellbauer (1980, p. 46). Their definition is that goods i and j are complements if $\partial h_i/\partial p_j$ is negative, and they are substitutes if $\partial h_i/\partial p_j$ is positive, where h_i is the Hicksian (or compensated) demand.

with a well-defined indirect utility function, $V_{ppq} > 0$ (or equivalently $x_{qp} < 0$). Since $V_{pq} < 0$ and $V_{ppq} > 0$, the functional form of V_q/V_y is U-shaped as in figure 1. With assumption (9), $x_q = 0$ at $p = \bar{p}$. Since $V_{qp} > 0$ for $p > \bar{p}$, x and q may be interpreted as substitutes in consumption beyond \bar{p} . This analysis implies that goods can be either complements or substitutes depending on the range of the private good price. In figure 1 if V_q/V_y is tangent to the horizontal axis at \bar{p} , then \bar{p} becomes equal to \bar{p} and there will be no difference between assumption (1) and assumption (9). However if V_q/V_y does not touch the price axis, we cannot get the threshold price with assumption (1). Figure 2 pictures two private good demand curves with different levels of a public good. Individual i 's willingness-to-pay for a public good is the shaded area in figure 2. The rightward shift of the demand curve is caused by the increase in the level of the public good provision (from q to q'). If the price of the private good is equal to \bar{p} , then there will be no change in the private good demand; i.e., $x_q(p, q, y) = 0$ at $p = \bar{p}$.

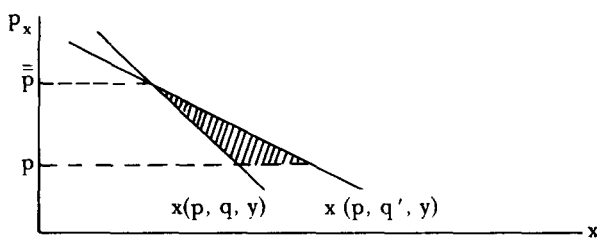


Figure 2

At a low price of the private good, demand for the good goes up as the provision of a public good increases ($x_q > 0$ at $p < \bar{p}$). As the price goes up, the degree of increase in the private good due to the same increase in the public good goes down ($x_{qp} < 0$). Finally at a high price there will be no change in the private good demand due to the increase in the public good ($x_q = 0$ at $p = \bar{p}$). And beyond that price the same increase in the public good reduces the demand for the private good. In this case incremental consumer's surplus can be maximized by setting the threshold price at \bar{p} . This maximized surplus is drawn as the shaded area. Any other price will lower the incremental consumer's surplus. The threshold price level is chosen with assumption (9). The numerical value of the threshold price depends on the specification of the private good demand function. If the

quantity demanded is zero at p' ($p' < \bar{p}$), then threshold price must be p' . If the private good demand is of the form of a hyperbolic function, both V_q and V_{qp} becomes zero at an infinite price level and quantity demanded will be zero at this price level.

Proposition 2 (Self-Interested Behavior): With restrictions (1) and (2), an individual's willingness-to-pay is captured with the Marshallian private good demand function in the interval between p and \bar{p} .

Proof: From Willing's Lemma 2 (1978, p. 241),

$$\frac{V_q(p, q, y)}{V_y(p, q, y)} = \int_p^{p'} x_q(\tau, q, y) d\tau + \frac{V_q(p', q, y)}{V_y(p', q, y)}$$

Let p' be equal to \bar{p} at which price a public good becomes valueless:

$$\frac{V_q(p, q, y)}{V_y(p, q, y)} = \int_p^{\bar{p}} x_q(\tau, q, y) d\tau + \frac{V_q(\bar{p}, q, y)}{V_y(\bar{p}, q, y)}$$

By assumption that $V_q(p, q, y) = 0$ at $p = \bar{p}$,

$$\frac{V_q(p, q, y)}{V_y(p, q, y)} = \int_p^{\bar{p}} x_q(\tau, q, y) d\tau.$$

Proposition 3 (Altruistic Behavior): With restrictions (8) and (9), an individual's willingness-to-pay is the change in Marshallian private good demand due to the change in the amount of a public good in the interval between p and \bar{p} .

Proof: From restriction (9),

$$\partial [V_q(p, q, y) / V_y(p, q, y)] / \partial p = 0 \quad \text{at } p = \bar{p}$$

The income compensation function, μ , is defined by

$$V\{p, q, \mu(p, q | p^\circ, q^\circ, y^\circ)\} = V(p^\circ, q^\circ, y^\circ) \quad (13)$$

This is the least income which achieves the same level of utility obtained in a base situation parameterized by $(p^\circ, q^\circ, y^\circ)$ when the consumer faces p and q . From (13),

$$V\{\bar{p}, q, \mu(\bar{p}, q | p^\circ, q^\circ, y^\circ)\} = V(p^\circ, q^\circ, y^\circ) \quad (14)$$

Differentiating (14) with respect to q gives

$$V_q\{\bar{p}, q, \mu(\bar{p}, q | p^\circ, q^\circ, y^\circ)\} + V_y(\bar{p}, q, \mu) \quad (15)$$

$$\mu_q(\bar{p}, q | p^0, q^0, y^0) = 0.$$

Let $\mu(\bar{p}, q | p^0, q^0, y^0) = y$. Then

$$\mu_q(\bar{p}, q | p^0, q^0, y^0) = - \frac{V_q(\bar{p}, q, y)}{V_y(\bar{p}, q, y)}. \quad (16)$$

Willing's Lemma 5 (1978, p. 243) states that restriction (8) is equivalent to

$$\mu_q(p, q) = - \int_p^\infty x_q\{\tau, q, \mu(p, q)\} d\tau. \quad (17)$$

Let $p = \bar{p}$. Then

$$\mu_q(\bar{p}, q) = - \int_{\bar{p}}^\infty x_q\{\tau, q, \mu(\bar{p}, q)\} d\tau = - \int_{\bar{p}}^\infty x_q(\tau, q, y) d\tau.$$

Note that since $\mu(p, q | p^0, q^0, y^0)$ is defined as y , $x\{p, q, \mu(p, q)\}$ is the Marshallian private good demand. From Willig's Lemma 2,

$$\begin{aligned} \frac{V_q(p, q, y)}{V_y(p, q, y)} &= \int_p^{\bar{p}} x_q(\tau, q, y) d\tau + \frac{V_q(\bar{p}, q, y)}{V_y(\bar{p}, q, y)} \\ &= \int_p^{\bar{p}} x_q(\tau, q, y) d\tau + \int_p^\infty x_q\{\tau, q, \mu(\bar{p}, q)\} d\tau \\ &= \int_p^{\bar{p}} x_q(\tau, q, y) d\tau + \int_{\bar{p}}^\infty x_q(\tau, q, y) d\tau \\ &= \int_p^{\bar{p}} x_q(\tau, q, y) d\tau. \end{aligned}$$

Note that in this case, the relevant upper price limit will be \bar{p} , because beyond \bar{p} the complementary relationship between the two goods will be changed and incremental consumer's surplus will be decreased. The efficiency condition will be

$$\sum_{i=1}^n \frac{V_q^i}{V_y^i} = \sum_{i=1}^n \int_p^{\bar{p}} x_q^i(\tau, q, y^i) d\tau = MC_q.$$

Note that the difference in individual i 's willingness-to-pay between the altruistic and self-interested behavior will be

$$\int_p^{\bar{p}} x_q^i(\tau, q, y^i) d\tau.$$

whose sign depends on the values of \bar{p} and \bar{p} .

This approach can be developed in several ways. First, even in the case of substitutes, the basic implications will not be changed. Once we can analyze the substitutes case, this approach will be applicable to a wide range of goods. Second, it is not necessary that one good be public and another be private. Even though a good belongs to the private good category, it is sometimes plausible to believe that we don't know the exact demand price (or willingness-to-pay) for that good. The approach described in this section can also be applied to this case.

Let's consider the substitute goods case. Assume that

$$V_q/V_y = 0 \quad \text{at} \quad p = \bar{p}, \quad \partial(V_q/V_p)/\partial V_y = 0$$

Two goods, x and q , are assumed to be substitutes if $x_q < 0$.

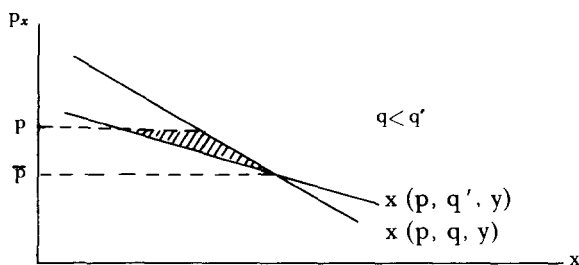


Figure 3

As we can see in figure 3, $x_q < 0$ at $p > \bar{p}$, $x_q = 0$ at $p = \bar{p}$ and $x_q > 0$ at $p < \bar{p}$. With the same indirect utility functional form (10),

$$- \int_{\bar{p}}^p x_q d\tau = - \int_{\bar{p}}^p \frac{\partial(-V_p/V_y)}{\partial q} d\tau = \frac{V_{pq}V_y - V_pV_{yq}}{V_y^2} d\tau.$$

Since $V_{qy} = 0$, we have

$$\begin{aligned} \int_{\bar{p}}^p \frac{V_{pq}V_y - V_pV_{yq}}{V_y^2} d\tau &= \int_{\bar{p}}^p \frac{V_{pq}}{V_y} d\tau \\ &= \frac{1}{V_y} [V_q(p, q, y) - V_q(\bar{p}, q, y)] = \frac{V_q(p, q, y)}{V_y(p, q, y)} \end{aligned}$$

Therefore willingness-to-pay will be

$$\frac{V_q}{V_y} = - \int_{\bar{p}}^p x_q(\tau, q, y) d\tau.$$

Note that the substitutive relationship will be changed below \bar{p} . The threshold price \bar{p} can be zero depending on the form of the private good demand function. The efficiency condition will be

$$\sum_{i=1}^n \frac{V_q^i}{V_y^i} = - \sum_{i=1}^n \int_{\bar{p}}^p x_q^i(\tau, q, y^i) d\tau = MC_q$$

Up to now we have analyzed the implications behind an indirect utility function which satisfies two different sets of assumptions. According to this analysis, what we have to do first is to construct an indirect utility function which satisfies relevant assumptions. The next step is to calculate the threshold price either from the indirect utility function or from the private good demand function. The final step is to sum individual willingness-to-pay and to equate this with the marginal cost of a public good and to solve for the efficient level of the public good. However enumerating these steps in this way is misleading. The major advantage of the weak complementarity approach lies in the relatively small amount of information we need to solve for the efficient level of a public good. Once we can set up the direct (or indirect) utility function, we no longer have to worry about the willingness-to-pay for the public good. The essence of the weak complementarity approach is that what we need is a private good demand function which is not independent of a public good. With some restrictions on the private good demand (thus on the implied indirect utility function), we can solve for the efficient level of a public good. In this sense following proposition has an important role.

Proposition 4: If a private good demand function exhibits constant income elasticity, then the indirect utility function has the form $V(p, q, y) = f(y) + g(p, q)$, which satisfies assumption (2).⁴⁾

Proof: Let η be income elasticity of the private good demand. Then

$$\frac{\partial x(p, q, y)}{\partial y} \cdot \frac{y}{x(p, q, y)} \equiv \eta.$$

4) The relationship between assumption (2) and income elasticity of the private good demand is as follows. The following three expressions are equivalent:

From Varian (1978, p. 211), the private good demand can be written as

$$x(p, q, y) = x(p, q, y^0) \left(\frac{y}{y^0} \right)^\eta.$$

$$\begin{aligned} \partial(V_q/V_p) / \partial y = 0 &\iff \partial \left[-(V_q/V_y) / (V_p/V_y) \right] / \partial y = 0 \\ &\iff \partial \left[(V_q/V_y) / x \right] / \partial y = 0. \end{aligned} \quad (4-1)$$

Substituting U_q/U_y for V_q/V_y in expression (4-1) gives

$$\begin{aligned} \partial \left(\frac{U_q/U_y}{x} \right) / \partial y = 0 \\ \iff (U_{qy}U_y - U_qU_{yy}) / x = U_qU_y (\partial x / \partial y). \end{aligned} \quad (4-2)$$

Multiplying both sides of (4-2) with (y/x) gives

$$(U_{qy}U_y - U_qU_{yy}) / y = U_qU_y (\partial x / \partial y) (y/x). \quad (4-3)$$

Note that: $(\partial x / \partial y) (y/x)$ is the income elasticity of the private good demand. Therefore,

$$\eta = \left(\frac{U_{qy}}{U_q} - \frac{U_{yy}}{U_y} \right) y = \frac{U_{qy}}{U_q} y - \frac{U_{yy}}{U_y} y. \quad (4-4)$$

The term, $-(U_{yy}/U_y)y$, in expression (4-4) is the form of relative risk aversion. Rewriting expression (4-4) gives

$$\eta = \frac{\partial U_q}{\partial y} \frac{y}{U_q} - \frac{\partial U_y}{\partial y} \cdot \frac{y}{U_y}. \quad (4-4)'$$

Note that the first term of the right-hand side expression is the income elasticity of the marginal utility of public good and the second term is the income elasticity of the marginal utility of income. Therefore if the second term depends on income, the income elasticity of private good demand is not constant.

Now let's go back to the indirect utility function expression. The second assumption is satisfied if the indirect utility function has the following form:

$$V(p, q, y) = f(y) + g(p, q) \quad (4-5)$$

The income elasticity of the private good demand, η , is

$$\begin{aligned} \eta = \frac{\partial x}{\partial y} \frac{y}{x} &= - \frac{\partial [g_q(p, q) / f_y(y)]}{\partial y} \frac{y}{x} = \left[- \frac{g_{pq}(p, q)}{f_y(y)} \right] \\ &\quad \left[- \frac{f_{yy}(y)}{f_y(y)} \right] = - \frac{f_{yy}(y)}{f_y(y)} y. \end{aligned} \quad (4-6)$$

Comparison of expression (4-4) which (4-6) shows that they are equivalent except for the first term of the right-hand side of expression (4-4). This term vanishes in expression (4-6) due to the form of indirect utility function specified in expression (4-5). Therefore the property of the income elasticity of private good demand depends on the functional form of $f(y)$. For example, homothetic preferences give

$$[f_y(y)]^{-1} = y \iff f_y(y) = \frac{1}{y} \iff f(y) = \ln y, \quad \eta = 1.$$

where y° is the base income. From Roy's identity,

$$x(p, q, y^\circ) \left(\frac{y}{y^\circ}\right)^\eta \frac{\partial V(p, q, y)}{\partial y} = - \frac{\partial V(p, q, y)}{\partial p}.$$

Now let's claim that $V(p, q, y) = f(y) + g(p, q)$ and check whether this claim is true:

$$\left(\frac{y}{y^\circ}\right)^\eta f_y(y) = - \frac{g_p(p, q)}{x(p, q, y)} \quad (18)$$

Expression (18) is satisfied for any non-negative y and p holding q fixed. Note that the left-hand side term of (18) is a function of y and the right-hand side is a function of p . Therefore each side must be equal to a constant. Let both sides be equal to 1. Then from the left-hand side,

$$f_y(y) = (y^\circ)^\eta (y)^{-\eta} \Rightarrow f(y) = (y^\circ)^\eta \int_0^y \omega^{-\eta} d\omega.$$

Note that $f(y)|_{y=0}$ is equal to zero due to zero utility. And from the right-hand side,

$$g_p(p, q) = -x(p, q, y^\circ) \Rightarrow g(p, q) = \int_p^\infty x(\tau, q, y^\circ) d\tau.$$

Note that $g(p, q)|_{p=\infty}$ is equal to zero due to zero utility. Therefore $f(y)$ is a function of y and $g(p, q)$ is a function of p and q . The claim turns out to be true.⁵⁾

The significance of this proposition is due to that fact that we can not

5) Proposition 4 can also be proved in the following way. Let's set aside the level of a public good, q , for a moment. The required steps are as follows:

Step 1: If a demand function has constant income elasticity, then an implied utility function has the form:

$$V(p, y) = g(p) + f(y),$$

which can find in Varian (1978, p. 229).

Step 2: If we include q as a parameter in step 1, then there are three possible indirect utility functional forms:

$$g(p, q) + f(y), \quad g(p) + f(q, y), \quad g(p, q) + f(q, y).$$

Step 3: In the presence of three possible forms, we can transform the latter two cases into the first one, through a strictly increasing, differentiable transformation of V . This transformation will generate the same demand function. For example, let

$$V(p, q, y) = g(p) + f(q, y) = p^{\alpha_1} + q^{\alpha_2} y^{\alpha_3}.$$

observe the marginal rate of substitution between q and y or equivalently the willingness-to-pay for a public good. From proposition 4 we know that the restriction on the private good demand function imposes a restriction on the implied indirect utility function. Initially two assumptions (8) and (9) were needed for analysis. So far a problem we faced was that these restrictions were expressed in terms of a utility function. However by dint of proposition 4 and the relationship between those restrictions, we escape this problems.

Now proposition 4 says if we have a constant-income-elasticity demand function, the implied indirect utility function obviously satisfies restriction (8). And restrictions (8) and (9) together imply that the partial derivative of the private good demand with respect to the level of the public good is equal to zero at \bar{p} , i.e., $x_q = 0$ at $p = \bar{p}$. Therefore the following sets of restrictions are equivalent:

$$\left. \begin{aligned} \partial (V_q/V_p) / \partial y &= 0 \\ \partial (V_q/V_y) / \partial p &= 0 \quad \text{at } p = \bar{p} \end{aligned} \right\} \leftrightarrow \left\{ \begin{aligned} \eta &\text{ is constant} \\ x_q &= 0 \quad \text{at } p = \bar{p}. \end{aligned} \right.$$

Therefore what we need for analysis are a private good demand function with constant-income-elasticity and the value of \bar{p} at which $x_q = 0$. The key point once again is that for the analysis we do not need a specific utility function which is unobservable but a private good demand function which is estimable.⁶⁾

$$\text{Then,} \quad X = \frac{V_q}{V_y} = -\frac{\alpha_1}{\alpha_3} p^{\alpha_1-1} q^{-\alpha_2} y^{1-\alpha_3}.$$

With the transformation of V ,

$$\therefore F(V) = F(p, q, y) = q^{-\alpha_2} V = p^{\alpha_1} q^{-\alpha_2} y^{\alpha_3},$$

where $F'(V) = q^{-\alpha_2} > 0$. Then

$$x = -\frac{F_p}{F_y} = -\frac{\alpha_1}{\alpha_3} F^{\alpha_1-1} q^{-\alpha_2} y^{1-\alpha_3}.$$

Therefore two functions V and F yield the same demand function. That means, if we have a demand function with constant income elasticity, we can assume an indirect utility function of the form:

$$V(p, q, y) = g(p, q) + f(y).$$

6) There is an interesting paper by Morey (1984) concerning this issue. According to him, there are two approaches to estimating demand equations. The one approach specifies an algebraic form for the utility function, derives the corresponding system of demand equations, and only then uses data to estimate the coefficients in those demand equations, imposing an algebraic form on the utility function with complete ignorance of data amounts unfortunately to imposing a priori

The following considerations are important before this approach can be applied. First, finite threshold prices with two individual behaviors (\bar{p}, \bar{p}) can be consistent with the properties of a well-behaved indirect utility function once the implicit assumption that the upper limit of the relevant price range with each individual behavior is a threshold price itself is adopted. Second, if the private good demand function is hyperbolic, then two threshold prices will be infinite. In this case we don't need an individual behavioral assumption, because the upper limit of the price will be infinite. Third, if the quantity of the private good demanded is zero at a finite price, then that price will be upper limit in calculating the amount of willingness-to-pay.

III. Summary and Suggested Further Work

An empirical analysis using this theoretical approach would be possible once we can select two related goods. An example would be a housing (private good) and the school quality (local public good). The demand for housing contains the information on the valuation of the local public good, education. Based on the analysis in section II, we can calculate a threshold price at which the partial derivative of the private good demand with respect to the level of a public good becomes zero. Behavioral assumptions (self-interested and altruistic) affect the level of a threshold price.

Once we construct the efficiency conditions for the public good, the efficient level of the public good provision can be calculated. The efficiency conditions are usually highly nonlinear, which results in a system of non-linear simultaneous equations for each local community. After calculating the efficient levels of the public good we can compare these with the actual levels. We can also test whether the discrepancy between the efficient and the actual levels of the public good is related to the characteristics of the communities. The "optimal" property tax rates to the communities with which the efficient levels of the public good are provided can be derived.

restrictions on the preference ordering. These a priori restrictions are embedded in the demand equation. Since their imposition was not based on the data, the estimated demand equations are not necessarily correct and the resulting ranking could be wrong. The other approach to demand estimation specifies the algebraic form for the demand equations directly and then uses the market data to estimate their coefficients. The algebraic form of the demand equations determines the form of the utility function up to a monotonic transformation once the demand equations satisfy the integrability condition (Hurwicz and Uzawa, 1971)-negative semidefiniteness of the substitution term matrices.

Computation of the efficient level of a public good based on the theoretical approach developed in this paper can be a reference point for the local government as it attempts to determine the level of a public good to provide, because this calculation is based on the preferences of residents in a community. Moreover, based on the empirical findings, with given expenditure levels, it may be possible for the local governments to provide the efficient levels of public goods by adjusting the mix of expenditures among all (or some) public goods. For example, if a community overspends on public education but underspends on other local public goods, then the local government may provide the efficient levels of all public goods by reallocating the expenditures.

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