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ON THE EXISTENCE OF A FINANCIAL EQUILIBRIUM IN ASSET TRADING ECONOMIES WHEN MARKET PARTICIPATIONS ARE RESTRICTED

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I. INTRODUCTION

When people form portfolios from financial assets for hedging against uncertain states in the future, it is common to observe that they are concerned only about a subset of assets available in the markets. Furthermore, different people may form their portfolios from different subsets of assets. That is, market participations in financial markets are restricted in general. This could be explained by the presence of transaction costs or by informational asymmetry, or by the imperfectness of financial markets in one word. But, it is surprising that most work about financial economies assumed full participations in or complete access to financial markets. Among few, Levy (1978) examined the effect of restricted participations on the implication of the "capital asset pricing model" in the finance literature. On the other hand, Siconolfi (1986) has shown the existence of a financial equilibrium with restricted participations as an extension of the general equilibrium model by Cass (1984) and Werner (1985). His analysis is strictly restricted to the case that the number of states is greater than the number of assets, i.e. that financial markets are incomplete.

But, if there are "too many" assets in the sense that some of them are *redundant* for hedging against uncertain states in the future, then people may be more easily inclined to participate only in the subsets of competitive financial markets. The objective of this paper is to examine the effect of restricted participations on the property of a financial equilibrium in a general setting, i.e. not only when financial markets are incomplete, but also when there are too many assets. A simple model of a financial economy is described in section II. Section III examines the behavior of budget hyperplanes and its relationship to the property of a financial equilibirum. Some concluding remarks are given in section IV.

II. THE MODEL

1. Overview

We consider a simple financial economy which lasts for two periods 0 and 1.

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There are uncertain states of nature in period 1, denoted by $s = 1,\ldots,N$. There is a finite number of consumers, indexed by $h \in H = \{1,\ldots,m\}$. In period 0, there is a spot market for trading commodities and financial assets. Also, there will be spot markets for commodities at each state in period 1. Thus, there are $(N + 1)$ spot markets in this economy so that we can identify $s \in S = \{0,1,\ldots,N\}$ as a "spot market". There are M financial assets available in the current financial markets, denoted by $f \in F = \{1,\ldots,M\}$, as instruments for transferring income across time and across states. Each asset "f" yields a fixed, state dependent return $r^f = (r^f(1),\ldots,r^f(N))$ in terms of unit of account. $q = (q^1,\ldots,q^M)$ represents the price vector of M assets which is supposed to be determined in the competitive financial markets. There are "L" commodities in each period, denoted by $c = 1,\ldots,L$. Each consumer's endowment of commodities is state contingent. It is denoted by a strictly positive vector $e_h = (e_h(0), e_h(1),\ldots,e_h(N)) \in R^{(N+1)L}_{++}$.

His consumption is denoted by $x_h = (x_h(0), x_h(1),\ldots,x_h(N)) \in X = R_+^{(N+1)L}$. $p = (p(0), p(1),\ldots,p(N))$ denotes the spot price vector of commodities which will be determined in the competitive markets. His preference under uncertainty is represented by a utility function $U_h : X \to R$ satisfying:

(A1) U_h is continuous.

(A2) U_h is strictly increasing and quasi–concave.

His problem is to choose the optimal consumption–portfolio plan by trading commodities and assets in spot markets.

2. Market Participation and Asset Prices without Arbitrage

In this economy, consumers may participate in the financial markets only in the restricted sense. To make this point clear, let [R] denote an $N \times M$ return matrix satisfying the following conditions[1]:

$$[R] = \begin{bmatrix} r^1(1),\ldots,r^M(1) \\ \vdots \qquad \vdots \\ r^1(N),\ldots,r^M(N) \end{bmatrix}$$

(A3) rank([R]) = full rank

(A4) $r^f(s) \geq 0$ for all $s = 1,\ldots,N$ and $f = 1,\ldots,M$

Now, let F_h denote a subset of F, denoting a collection of assets in which markets consumer h wants to participate. $\#(F_h)$ denotes the cardinality of F_h. Then, $[R_h]$ denotes a submatrix of [R], composed of column vectors of returns from assets in F_h. $\{F_h\}$ denotes the participation pattern chosen by consumers, satisfying the following conditions[2]:

[1] There is no loss of generality in imposing these conditions since the set of $N \times M$ matrices satisfying these conditions is open and dense in the set of all $N \times M$ matrices with non–negative elements.

(A5) $\#(F_h) \leq M$ for every $h \in H$. rank($[R_h]$) = full rank and each row of $[R_h]$ is non–trivial.

(A6) For every $f \in F$, there are at least two consumers $i, j \in H$ such that $f \in F_i \cap F_j$.

(A5) implies that each consumer will participate in financial markets, considering the possibility of income transfer at every state. (A6) implies that every asset market is active with $\bigcup_{h \in H} F_h = F$. Note that we do not impose any condition on the relationship between "N and M" so far.

Next, let B_h denotes consumer h's set of portfolio holdings according to his participation with a generic element $b_h \in R^M$. Since short sales are allowed here, B_h is a subspace of R^M such that:

$$B_h = \{b_h \in R^M : B_h^f = 0 \text{ for } f \notin f_h\} \tag{1}$$

When q is given in the financial markets, $[R_q]$ denotes an $(N+1) \times M$ augmented prices–returns matrix such that:

$$[R_q] = \begin{bmatrix} -q^T \\ \\ R \end{bmatrix}$$

where q^T is the transpose of q.[3]

Then, we can define $Y_h(q)$ as the set of income transfers relative to the given $[R_q]$:

$$Y_h(q) = \{y_h \in R^{N+1} : y_h = [R_q]b_h \text{ for some } b_h \in B_h\} \tag{2}$$

which is a subspace of R^{N+1}, parameterized by q. Note that there can be some $b_h \in B_h$ such that $y_h \geq 0$ with at least one strictly positive coordinate, which implies arbitrage opportunities unless q is restricted to some subset of R^M. Let q_h be the collection of asset prices in accordance with F_h and let q_{-h} be the collection of other asset prices. Consumer h is concerned only about q_h. It is assumed that $q_h * q_{-h} = q$ for the notational convenience. Define the following set:

$$Q_h = \{q \in R^M: \text{there is } \theta \in R^N_+ \text{ such that } q_h^T = \theta^T[R_h]\} \text{ for } h \in H$$
$$Q_F = \{q \in R^M : \text{there is } \theta \in R^N_+ \text{ such that } q^T = \theta^T[R]\} \tag{3}$$

Q_h and Q_F are obviously convex cones. Let Q_h^+ and Q_F^+ denote the interior of Q_h and Q_F respectively. Define $Q = \bigcap_{h \in H} Q_h$.

Then, $Q^+ = (\bigcap_{h \in H} Q_h)^+ = \bigcap_{h \in H} Q_h^+$. Observe that Q^+ is the set of no–arbitrage asset prices relative to $\{F\}$ while Q_F^+ is the one relative to full participations by

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[2]There is also no loss of generality in adding (A5) since almost all [R] satisfying (A3) and (A4) satisfies also (A5) no matter what $\{F_h\}$ is chosen.

[3]Any vector with the superscript "T" denotes a row vector and hence any one without "T" denotes a column vector.

the application of the Minkowski–Farkas lemma.[4] Since Q_F is a subset of Q_h for every h, Q_F is also a subset of Q and hence Q_F^+ is a subset of Q^+. Note that Q is also a convex cone as a finite intersection of convex cones and hence Q^+ is the interior of a convex cone. The structure of Q^+, including Q_F^+ always as its subset regardless of $\{F_h\}$, is related to the behavior of budget hyperplanes and to the existence of a financial equilibrium in this economy. The detailed discussion will be deferred to the next section.

III. THE EXISTENCE AND OPTIMALITY OF A FINANCIAL EQUILIBRIUM

1. The Behavior of Budget Hyperplanes in Financial Economies with Restricted Participations

One important feature of a financial economy with restricted market participation is that consumers budget hyperplanes are possibly different subspaces of the ambient space. For the further discussion, let $z_h = (x_h - e_h) \in R^{(N+1)L}$ denote consumer h's excess demand vector and $z = (z_1, \ldots, z_m)$. Also, $b = (b_1, \ldots, b_m)$. Let [p] denote an $(N+1) \times (N+1)L$ matrix of spot prices such that:

$$[p] = \begin{bmatrix} p(0)^T & 0 \cdots\cdots & 0 \\ 0 & p(1)^T \; 0 \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots\cdots\; 0 & p(N)^T \end{bmatrix}$$

Then, each consumer will choose his optimal (z_h, b_h) relative to the given (p,q):

$$\begin{aligned} &\text{Maximize } U_h\,(z_h + e_h) \\ &\text{subject to } [p]\,z_h = [R_q]\,b_h \\ &\qquad \text{and } b_h \in B_h \end{aligned} \tag{4}$$

Since it is interesting to examine how z_h behaves relative to F_h, let's define the following sets:

$$Z_h(p, q) = \{(z_h, b_h) \in R^{(N+1)L} \times B_h : [p]\,z_h = [R_q]\,b_h\} \tag{5}$$

$$Z^P{}_h\,(p, q) = \{z_h \in R^{(N+1)L} : (z_h, b_h) \in Z_h(p, q)\} \tag{6}$$

Also, define the following sets relative to $q \in Q^+$:

$$T^f(q) = \{\theta \in R^N : q^f = \theta^T r^f\} \text{ for each } f \in F \tag{7}$$

$$T_h(q) = \{\theta \in R^N : q_h^T = \theta^T [R_h]\} \text{ for every } h \in H \tag{8}$$

$$\text{Then, by definition, } T_h(q) = \bigcap_{f \in F_h} T^f(q) \tag{9}$$

$$\text{Let } T(q : H) = \bigcap_{h \in H} T_h(q) \text{ and } T(q : F) = \bigcap_{f \in F} T^f(q) \tag{10}$$

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[4]See D. Gale (1960) for instance.

Lemma 1.
$T(q : H) = T(q : F)$.

PROOF: From (9) and (10), $T(q : H) = \bigcap_{h \in H} (\bigcap_{f \in F_h} T^f(q)) = \bigcap_{f \in F} T^f(q) = T(q : F)$. Q.E.D.

Notice that $T(q : H)$ may be empty. Whether it is empty or not depends upon the participation pattern $\{F_h\}$ and the relationship between N and M. This is crucial for the examination of the behavior of $Z_h(p, q)$ and hence for the existence of a financial equilibrium. To analyze this point in detail, it is very useful to introduce Arrow securities with a slight modification.

A Financial Economy with Arrow Securities

There are "N" Arrow contingent securities. Arrow security "s" costs π^s units of account and pays off one unit of account if and only if a state "s" occurs in period 1. Thus, the prices–returns can be represented by an $(N + 1) \times N$ matrix of the following form:

$$[I_\Pi] = \begin{bmatrix} -\pi^T \\ \\ I_N \end{bmatrix}$$

where I_N is an $N \times N$ identity matrix and $\pi = (\pi^1, \ldots, \pi^N)$. Let A_h denote consumer h's set of portfolios of Arrow securities with a generic element $a_h = (a^1{}_h, \ldots, a^N{}_h)$. A_h is assumed to be restricted as follows:

(A7) $A_h = Sp([R_h])$

where $Sp([R_h])$ is the subspace of R^N, spanned by the column vectors in $[R_h]$. So, $A = Sp([R])$. Let $\{A_h\}$ denote the collection of A_h for all h.

A consumer's problem in this economy is as follows:

$$\begin{aligned} &\text{Maximize } U_h(z_h + e_h) \\ &\text{subject to } [p]z_h = [I_\Pi]a_h \\ &\qquad \text{and } a_h \in A_h \end{aligned} \tag{11}$$

Here, there may be some arbitrage opportunities without any restriction on $\pi \in R^N$. For this, let $D(A_h)$ denotes the orthogonal complement of A_h. $D(A)$ is also defined in the same way. It is obvious that $D(A) \subseteq D(A_h)$ for all h. Let's define the following sets:

$$T(A : \pi) = D(A) + \{\pi\} \text{ and } T(A_h : \pi) = D(A_h) + \{\pi\} \text{ for some } \pi \in R^N \tag{12}$$

$$G_I = \{\pi \in R^N : T(A : \pi)R^N_{++} \neq \phi\} \tag{13}$$

$$G_2 = \{\pi \in R^N : T(A_h : \pi) \cap R^N_{++} \neq \phi \text{ for all } h \in H\} \tag{14}$$

Lemma 2.

G_2 is an open and convex set containing G_1 and hence R^N_{++} as its subset relative to any $\{A_h\}$.

PROOF: First, it is obvious that $R^N_{++} \subsetneq G_1$ by the definition of $T(A : \pi)$. Since $D(A) \subsetneq D(A_h)$, $T(A : \pi) \subsetneq T(A_h : \pi)$ for all h. Thus, if $\pi^* \in G_1$, $T(A_h : \pi^*) \cap R^N_{++} \neq \phi$ for all h. Hence, $\pi^* \in G_2$, implying that $R^N_{++} \subsetneq G_1 \subsetneq G_2$.

Next, G_2 is obviously an open set. Now, suppose that π, $\pi' \in G_2$ such that $\pi \neq \pi'$. Let $\pi'' = \mu\pi + (1-\mu)\pi'$ for $\mu \in (0, 1)$. Define $T(A_h : \pi'')$ for every $h \in H$. Since $T(A_h : \pi'')$ is parallel to $T(A_h : \pi)$ and $T(A_h : \pi')$, there is some $V_h \in (0, 1)$ such that $\delta_h = v_h\alpha_h + (1 - v_h)\beta_h \in T(A_h : \pi'') \cap R^N_{++}$ where $\alpha_h \in T(A_h : \pi) \cap R^N_{++}$ and $\beta_h \in T(A_h : \pi') \cap R^N_{++}$ for every $h \in H$. Thus, $\pi'' \in G_2$ and hence G_2 is convex. Q.E.D.

Remark 1.

Notice that G_2 is the set of no-arbitrage Arrow securities prices when A_h is given as the set of portfolios of consumer h. Suppose that $\mu \in T(A_h : \pi)$ for some π with $T(A_h : \pi) \cap R^N_{++} = \phi$. Since $\mu = d + \pi$ for $d \in D(A_h)$, $(1, \mu)^T y_h = (1, \mu)^T [I_{II}] a_h = -\sum_s \pi^s a^s_h + \sum_s (d^s + \pi^s) a^s_h = 0$. So, there can be $a_h \in A_h$ such that $y_h = [I_{II}] a_h \geq 0$ with at least one strictly positive coordinate. Also, observe that $\pi \in G_2$ is not necessarily a strictly positive vector unless there is some A_h such tht $A_h = R^N$.

Now, go back to the original economy and examine the property of $T(q : H)$. It is easy to see that $T(q : H)$ is always non-empty for each $q \in Q^+$ if there is at least one consumer j with full participation, i.e. $\#(F_j) = M$.

So, let's consider what happens if (A5) is replaced by (A8):

(A8) $N \geq M$ and $\#(F_h) < M$ for all $h \in H$. rank($[R_h]$) = full rank and each row of $[R_h]$ is non-trivial.

Lemma 3.

Suppose that (A8) holds. Then, $T(q : H)$ is always non-empty for every $q \in Q^+$.

PROOF: From the definition of Q^+ and G_2, and from (A3), (A7) and (A8), there is a surjective linear mapping $\Gamma : G_2 \rightarrow Q^+$ such that $\Gamma(\pi) = \pi^T[R] = q \in Q^+$. Hence $\Gamma^{-1}(q) \subsetneq G_2$ is non-empty for every $q \in Q^+$. Since $\Gamma^{-1}(q) \subsetneq T(q : F)$ and $T(q : F) = T(q : H)$ by lemma 1, the proof is completed. Q.E.D.

Now, let's define $Y_h(\pi)$ in accordance with $Y_h(q)$:

$Y_h(\pi) = \{y_h \in R^{N+1}$: there is $a_h \in A_h$ such that $y_h = [I_{II}] a_h\}$ relative to some $\pi \in G_2$. (15)

Let $(Y_h(q), Q^+)$ and $(Y_h(\pi), G_2)$ denote alternative representations of financial opportunities.

Definition 1.

$(Y_h(q), Q^+) \approx (Y_h(\pi), G_2)$ denotes identical financial opportunities in the sense that there is always some $\pi \in G_2$ such that $Y_h(q) = Y_h(\pi)$ for some $q \in Q^+$ for every $h \in H$.

PROPOSITION 1: Suppose that (A8) holds. Then, $(Y_h(q), Q^*) \approx (Y_h(\pi), G_2)$.

PROOF: By lemma 3, there is always some $\theta^* \in T(q : H)$ for $q \in Q^+$. Set $\pi = \theta^*$. Then, it is obvious that $\pi \in G_2$ from the definition of G_2 and $T(q : H)$. Furthermore, pick some $y_h \in Y_h(q)$ such that $y_h(0) = -q^T b_h$ and $y_h(s) = r(s)^T b_h$ for $s = 1,..,N$. Set $[R] b_h = a_h \in A_h$ for every h by (A7). Since $q^T = \theta^{*T}[R]$, $-q^T b_h = -\theta^{*T}[R] b_h = -\pi^T a_h$. Thus, $y_h \in Y_h(\pi)$ for some $\pi \in G_2$. Q.E.D.

The equivalence of these two representations is very useful for the further discussion. So, let's discuss the behavior of budget hyperplanes in terms of $Y_h(\pi)$. Again, let's define consumers' budget hyperplanes relative to p and $\pi \in G_2$:

$$Z_h(p, \pi) = \{(z_h, a_h) \in R^{(N+1)L} \times A_h : [p] z_h = [I_\Pi] a_h\} \tag{16}$$

$$Z_h^P(p, \pi) = \{z_h \in R^{(N+1)L} : (z_h, a_h) \in Z_h(p, \pi)\} \tag{17}$$

$$Z^W(p_\Pi) = \{z_h \in R^{(N+1)L} : (1, \pi)^T [p] z_h = 0\} \tag{18}$$

Let's call $Z^W(p_\Pi)$ as the Walrasian budget hyperplane with a normal vector $p_\Pi = (p(0), \pi^1 p(1),.., \pi^N p(N))$.

PROPOSITION 2: If (A8) holds, $Z_h^P(p,q)$ is a subspace of $Z^w(p_\Pi)$ for $\pi \in G_2$ and for every $h \in H$ when $q^T = \pi^T[R]$ with the property that $Z_i^P(p,q) = Z_j^P(p,q)$ if $Sp([R_i]) = Sp([R_j])$.[5]

PROOF: From $[p] z_h = [I_\Pi] a_h = y_h$, $(1, \pi)^T [p] z_h = (1, \pi)^T [I_\Pi] a_h = (1, \pi)^T y_h = 0$ for any $z_h \in Z_h^P(p, \pi)$. Since $Z_h^P(p, \pi) = Z_h^P(p,q)$ when $q^T = \pi^T[R]$ for $\pi \in G_2$, obviously $Z_h^P(p,q)$ is a subspace of $Z^w(p_\Pi)$ for every $h \in H$. Q.E.D.

Remark 2

There are many subspaces of dimension $[(N+1)L-1]$, containing $Z_h^P(p,q)$ rela-

[5]This is the general equilibrium analogy to the relationship among "efficient sets" of the CAPM in Levy (1978) when there is a constraint on the number of securities in protfolios.

tive to $(p,q) \in R_{+}^{(N+1)L} \times Q^{+}$.[6] This collection of subsapces is different for different consumers according to F_h. But, there is always at least one subspace of dimension $[(N+1)L-1]$, orthogonal to p_{II}, which contains every $Z_h^P(p,q)$ for $\pi \in G_2$ such that $q^T = \pi^T[R]$. Thus, it can be said that the financial economy here is the analogy to Arrow–Debreu economy in the following sense:

Every consumer's budget hyperplane is always a subset of the Walrasian budget hyperplane and we can always restrict our attention to the consumption demand correspondence because commodities markets clearing implies assets markets clearing by Walras' law and (A3). The only difference is that the set of net trades for every consumer is always a subset of Walrasian set of net trades.

Next, consider the case that (A8) is replaced by (A9):

(A9) $M > N$ and $\#(F_h) \leq M$. rank$([R_h])$ = full rank and each row of $[R_h]$ is non–trivial.

This implies that some assets are *redundant* so that consumers are easily inclined to restricted participations. Let $W([R]) = \{q \in R^M :$ There is $\theta \in R^N$ such that $q^T = \theta^T[R]\}$.

PROPOSITION 3: Suppose that (A9) holds. Then, $Z_h^P(p,q)$ is subspace of $Z^w(p_{II})$ for every $h \in H$ with $q^T = \pi^T[R]$ if and only if $q \in Q^+ \cap W([R])$.

PROOF: Now, it is easy to see that $T(q:H) \neq \phi$ if and only if $q \in Q^+ \cap W([R])$. Then, using the definition of G_2 and the definition 1, we can obtain the desired result. Q.E.D.

Thus, it is impossible to represent financial opportunities in the original economy by Arrow securities when q varies out of $Q^+ \cap W([R])$. So, the analogy to Arrow–Debreu economy may hold only when q lies in $Q^+ \cap W([R])$ and $Q^+ \cap W([R])$ is determined by the structure and dimension of Q^+. Finally, note that the structure and dimension of Q^+ is determined by the pattern $\{F_h\}$. For this, we can make the following observations:

Observation 1.

Suppose that (A9) holds and $\#(F_h) < N$ for all h. Then, Q^+ is a non–empty, open subset in R^M and $Q_F^+ \subseteq Q^+ \cap W([R])$.

PROOF: It is obvious that Q_h^+ is an open subset in R^M, containing Q_F^+ for every $h \in H$. Since a finite intersection of open sets is also an open set, $Q^+ = \bigcap_{h \in H} Q_h^+$ is a non–empty, open set in R^M. Since $Q_F^+ \subseteq Q^+$ and also $Q_F^+ \subseteq W([R])$ by definition, $Q_F^+ \subseteq Q^+ \cap W([R])$. Q.E.D.

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[6]By the definition of $T_h(q)$, the dimension of $T_h(q) = N - \#(F_h)$. There is a distinctive $[(N+1)L]-1$ – dimensional budget hyperplane corresponding to each $\theta \in T_h(q)$, containing $Z_h^P(p,q)$.

Observation 2.

Suppose that (A9) holds and that there is at least one consumer $j \in H$ with $N \leq \#(F_j) \leq M$. Then $Q^+ \cap W([R]) = Q_F{}^+$.

PROOF: It holds still that $Q_F{}^+ \subseteq Q^+ \cap W([R])$. Now, suppose that there is some $q \in Q^+ \cap W([R])$ such that $q \notin Q_F{}^+$. Then, there is a unique $\theta \in R^N \setminus R^N{}_{++}$ such that $q^T = \theta^T[R]$ by definition. Moreover, $q_j{}^T = \theta^T[R_j]$ holds, which implies arbitrage opportunities for consumer j. This is a contradiction. Q.E.D.

Thus, if (A9) holds, financial opportunities will be represented completely by Arrow securities only if some particular $\{F_h\}$ is chosen.

PROPOSITION 4: Suppose that (A9) holds and there is at least one consumer $j \in H$ such that $\#(F_j) = M$. Then, Z_h^P (p,q) is a subspace of $Z^W(p_{II})$for every $h \in H$ with $q^T = \pi^T[R]$ for any $q \in Q^+$.

PROOF: It is clear that $Q^+ = Q^+ \cap W([R]) = Q_j{}^+ = Q_F{}^+$. Thus $T(q : H) = T_j(q) \neq \phi$ for every $q \in Q^+$. Pick some $\theta \in T(q : H)$ and set $\theta = \pi$. Then, the proof will be completed by the same steps as in proposition 1 and 2. Q.E.D.

Remark 3

As was pointed out in Remark 2, the consumption demand correspondence for every h can be thought of as the analogy to that in the Arrow–Debreu economy if (A8) holds. So, we can apply the fixed point argument for the existence of a financial equilibrium as will be seen in the next section. But, if (A9) holds, then this analogy holds no longer and hence there is no obvious way of applying the fixed point argument to show the existence of a financial equilibirum.

2. Market Participations and Existence of a Financial Equilibirum

It will be shown here that the existence of a financial equilibrium is related to the behavior of budget hyperplanes discussed in the previous section.

Definition 2.

A collection (p,q,z,b) is a Restricted Financial Equilibrium (RFE) if 1) $(z_h, b_h) \in Z_h(p,q)$ and there is no $(z_h{}', b_h{}') \in Z_h(p,q)$ such that $U_h(z_h{}'+e_h) > U_h(z_h + e_h)$ for all $h \in H$ and 2) $\sum_{h \in H} z_h = 0$, $\sum_{h \in H} b_h^f = 0$ for all $f = 1,..,M$.

Definition 3.

A collection (p, π, z, a) is Restricted Arrow Equilibrium (RAE) if 1) $(z_h, a_h) \in Z_h(p, \pi)$ and there is no $(z_h{}', a_h{}') \in Z_h(p, \pi)$ such that $U_h(z_h{}' + e_h) > U_h(z_h + e_h)$ for all $h \in H$ and 2) $\sum_{h \in H} z_h = 0$, $\sum_{h \in H} a_h^s = 0$ for all $s = 1,..,N$.

As the equivalence of alternative representation of financial opportunities is useful in examining the behavior of budget hyperplanes, it is useful to compare these two equilibirum concepts.

First, observe that G_2 contains R^N_{++} regardless of the relationship between N and M and of the pattern $\{F_h\}$ by lemma 2. So, we can use the following price sets with "state by state" normalization:

$$P(0) = \{(p(0),\pi) \in R^L_+ \times R^N_+ : \sum_c p^c(0) + \sum_s \pi^s = 1\}$$
$$P(s) = \{p(s) \in R^L_+ : \sum_c p^c(s) = 1\} \text{ for } s = 1,..,N$$
$$P = P(0) \times P(1) \times ..\times P(N)$$

PROPOSITION 5: Under the assumptions about endowments, preferences and restrictions on portfolio choices (A7), there exists a RAE (p^*,π^*,z^*,a^*) whether (A8) or (A9) holds.

PROOF: Here, the sektch of proof is given in line with the argument in Werner (1985). First, let's define the following sequence of price sets:

$$P(0)^v = \{(p(0),\pi) \in P(0) : p^c(0) \geq 1/v \text{ and } \pi^s \geq 1/v \text{ for all } c = 1,..,L \text{ and } s=1,..,N\}$$
$$P(s)^v = \{p(s) \in P(s) : p^c(s) \geq 1/v \text{ for all } c = 1,..,L \text{ and } s = 1,..,N\}$$
$$\text{where } v > N + L$$

Then, the demand correspondence $(z_h(p,\pi), a_h(p,\pi))$ is nonempty, compact, convex and upper semi-continuous at every $(p,\pi) \in P^v$.[7] Let K^v be a nonempty, compact and convex set in $R^{(N+1)L} \times R^N$ such that $(z_h(p,\pi), a_h(p,\pi)) \in K^v$ for $(p,\pi) \in P^v$, for every h = 1,..,m. Now, define the following correspondence $\tau^v = \phi^v \times \mu^v: p^v \times (K^v)^n \rightarrow P^v \times (K^v)^n$ such that:

$$\phi^v(p,\pi) = \{(z_h(p,\pi), a_h(p,\pi)) \in K^v \text{ for } h = 1,..,m\} \text{ and}$$
$$\mu^v(z,a) = \{p \in P^v:(p(0),\pi) = \text{argmax}[p(0)' \sum_{h\in H} z_h(0) + \sum_s \pi^{s'} \sum_{h\in H} a^s_h] \text{ and}$$
$$p(s) = \text{argmax}[p(s)' \sum_{h\in H} z_h(s)] \text{ for } s=1,..,N\}$$

Since τ^v is easily identified to be a upper semi-continuous correspondence from a nonempty, convex and compact set into itself and $\tau(p,\pi,z,a)$ is also compact, we can apply Kakutani's fixed point theorem to get $(p^v,\pi^v,z^v,a^v) \in \tau^v(p^v,\pi^v,z^v,a^v)$. As $v \rightarrow \infty$, (p^v,π^v,z^v,a^v) converges to $(p^*,\pi^*,z^*,a^*) \in \tau^*(p^*,\pi^*,z^*,a^*)$ which satisfies the conditions in definition 3. Q.E.D.

Remark 4.

The existence of a RAE has been shown in a general setting, but this does not always ensure the existence of a RFE. If all the maintained assumptions hold

[7]See Werner (1985) or Siconolfi (1986) for the complete description about the behavior of the demand correspondence in financial economies, especially about its behavior on the boundary of price simplices. The detailed explanation is skipped here to avoid any unnecessary exposition.

together with (A8), then there exists a RFE *if and only if* a RAE exists because $[R](\sum_{h\in H} b_h) = \sum_{h\in H} a_h = 0$ implies that $\sum_{h\in H} b_h = 0$ due to (A3). But, if (A8) is replaced by (A9), then the existence of RAE does not guarantee the existence of a RFE. Again, $\sum_{h\in H} a_h = 0$ does not necessarily imply $\sum_{h\in H} b_h = 0$ due to (A3).

Remark 5.

If (A8) describes the relationship between N and M and the participation pattern, we can restrict our attention to $Q_F{}^+$ instead of Q^+ for the existence of a RFE because restricting π to R^N_{++} is equivalent to resticting q to $Q_F{}^+$.

Although restricted participations allow a larger set of no–arbitrage asset prices than full participations, it is not necessary for us to consider all $q \in Q^+$ for the existence of a RFE per se.

Remark 6.

Define the following set: $V = \{\mu = (\mu^1, ..., \mu^N) \in R^N_{++} : \mu^1 = 1\}$ Then, any RFE (p,q,z,b) with the return structure $[R'] = [\mu][R]$ will be supported as a RFE (p′,q′,z′,b′) with [R] such that $p' = (p(0), (1/\mu^1)p(1),...,(1/\mu^N)p(N))$, $q' = q$, $z' = z$ and $b' = b$ where $[\mu]$ is a diagonal matrix with $\mu \in V$ on the main diagonal. This is the extension of what Werner (1985) and Geanakoplos and Mas–Colell (1985) pointed out in the case of full participation. But, it remains to be seen whether there is real indeterminacy of RFE allocations regardless of $\{F_h\}$. See Lee (1987) for the discussion about this problem.

Next, consider the existence of a RFE when (A9) describes the participation pattern with some redundant assets. As was mentioned in Remark 3, the analogy to Arrow–Debreu economy holds no longer in general. This implies that we can not restrict our attention to the consumption demand correspondence even when q varies in $Q^+ \cap W([R])$. Moreover, when q varies out of $Q^+ \cap W([R])$, the behavior of consumption–portfolio demand correspondence becomes more complicated and there is no obvious way of applying the fixed point argument for the existence of a RFE.

Let's take a look at the following heuristic example for understanding the problem about the property of a RFE in this case.

[Example]

Suppose that there are two uncertain states, $s = \alpha, \beta$ in period 1. There are 3 assets with the following return structure:

$$[R] = \begin{bmatrix} r^1(\alpha), r^2(\alpha), r^3(\alpha) \\ r^1(\beta), r^2(\beta), r^3(\beta) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

So, there are two state contingent Arrow securities and an inside money in financial markets.

There is one commodity in each period 0 and 1. There are 3 consumers h = 1, 2, 3 in this economy and they are assumed to participate in financial markets as follows:

$\#(F_h) = 2$ for every h = 1, 2, 3 and there are exactly 2 consumers in each asset market, f = 1, 2, 3.

Since $A_h = Sp([R_h]) = R^2$ for all h, we can make the following observation: Restrict q to $Q_F^+ \subseteq Q^+$.

Then, $(Y_h(q)), Q_F^+) \approx (Y_h(\pi), G_2)$ where $G_2 = R^2_{++}$ by proposition 3 and observation 2. By proposition 5, there exists a RAE (p, π, z, a). Furthermore, it is Pareto optimal since $A_h = R^2$ for all h and there is nominal indeterminacy in the space of security prices and portfolios.

To see this, let (p^*, π^*, z^*, a^*) be a Pareto optimal RAE with some $\pi^* \in R^2_{++}$ such that $\pi^{\alpha *} + \pi^{\beta *} = 1$. Then, (p, π, z, a) is a Pareto optimal RAE with the following property:

$$p = (p^*(0), \pi^{\alpha *}p^*(\alpha), \pi^{\beta *}p^*(\beta)), \quad \pi = (1/\pi^{\alpha *}, 1/\pi^{\beta *}),$$

$$a_h = (\pi^{\alpha *}a_h^{\alpha *}, \pi^{\beta *}a_h^{\beta *}) \text{ with } z_h = z_h^* \text{ for all h.}$$

This implies that every $\{a_h\}$ supporting a Pareto optimal z^* is comletely characterized by varying $\pi^* \in R^2_{++}$ with the fixed $\{a_h^*\}$.

Now, suppose that the endowment profile $e_h \in R^3_{++}$ for h = 1,2,3 satisfies the following property:

$a_1^{\alpha *}, a_2^{\alpha *} > 0$ (and hence $a_3^{\alpha *} < 0$) and $a_2^{\beta *}, a_3^{\beta *} > 0$ (and hence $a_1^{\beta *} < 0$). This specification of $\{a_h^*\}$ describes the direction of net trades in the economy with Arrow securities. Now, examine the possibility of achieving this Pareto optimal allocation $z_h^* = x_h^* - e_h$ for all h as a RFE in the original economy. Since the existence of a RFE may depend upon the participation pattern $\{F_h\}$ chosen by consumers, it is necessary to make a complete list of all $\{F_h\}$ satisfying the above conditions:

$$\{F_h\}^1 = \{F_1 = (1,3),\ F_2 = (1,2),\ F_3 = (2,3)\}$$
$$\{F_h\}^2 = \{F_1 = (1,2),\ F_2 = (2,3),\ F_3 = (1,3)\}$$
$$\{F_h\}^3 = \{F_1 = (2,3),\ F_2 = (1,3),\ F_3 = (1,2)\}$$
$$\{F_h\}^4 = \{F_1 = (1,2),\ F_2 = (1,3),\ F_3 = (2,3)\}$$
$$\{F_h\}^5 = \{F_1 = (1,3),\ F_2 = (2,3),\ F_3 = (1,2)\}$$
$$\{F_h\}^6 = \{F_1 = (2,3),\ F_2 = (1,2),\ F_3 = (1,3)\}$$

when $\{F_h\}$ is chosen, there is a Pareto optimal RFE (p,q,z,b) if and only if there is a RAE (p, π, z, a) with the following property : $q^T = \pi^T[R]$, $[R]b_h = a_h$ such that $\Sigma b_h^f = 0$ for f = 1,2,3. But, $\Sigma a_h = 0$ at a RAE implies that asset markets clear if one of them, say f = 1, clears. So, let $B^1(t)$ denote the aggregate excess demand for f = 1 when $\{F_h\}^t$ is chosen for t = 1,..,6. Then, using the relation $[R]b_n = a_h$ and $a_h = (\pi^{\alpha *}a_h^{\alpha *}, \pi^{\beta *}a_h^{\beta *})$, we can get the following equations:

$B^1(1) = (a_1^{\alpha *} + a_2^{\alpha *})\pi^{\alpha *} - a_1^{\beta *}\pi^{\beta *} = 0$ $B^1(2) = (a_1^{\alpha *}+a_3^{\alpha *})\pi^{\alpha *} - a_3^{\beta *}\pi^{\beta *} = 0$

$B^1(3) = (a_2^{\alpha *} + a_3^{\alpha *})\pi^{\alpha *} - a_2^{\beta *}\pi^{\beta *} = 0$ $B^1(4) = (a_2^{\alpha *}+a_1^{\alpha *})\pi^{\alpha *} - a_2^{\beta *}\pi^{\beta *} = 0$

$B^1(5) = (a_1^{\alpha *} + a_3^{\alpha *})\pi^{\alpha *} - a_1^{\beta *}\pi^{\beta *} = 0$ $B^1(6) = (a_3^{\alpha *}+a_2^{\alpha *})\pi^{\alpha *} - a_3^{\beta *}\pi^{\beta *} = 0$

From the assumption on endowment profile, it is easy to see that $B^1(t) = 0$ has strictly positive solutions $\pi^* = (\pi^{\alpha *}, \pi^{\beta *}) \in R^2_{++}$ when $t = 4,5$. This implies that there exists a Pareto optimal RFE (p,q,z,b) if and only if $\{F_h\}^4$ or $\{F_h\}^5$ is chosen by consumers. So, if some $\{F_h\}$, $\{F_h\}^2$ is chosen, then there is no Pareto optimal RFE, but this does not imply that there does not exist a RFE at all. There may exist a RFE (p,q,z,b) with $q \in Q^+ \backslash Q_F^+$. But, this can't be Pareto optimal (this point is more clear in differentiable framework). So, we can make the following remark from this example:

Remark 7

Although each consumer forms his portfolios from sufficient number of assets in the sense that financial markets are complete from individual viewpoint, i.e. $\#(F_h) = N$ for every h, one of the following cases will hold relative to [R], $\{F_h\}$ and endowment profile:

1) There may not exist a RFE (p,q,z,b) whether it is Pareto optimal or not.
2) There may not exist a Pareto optimal RFE (p,q,z,b), but some (p′,q′,z′,b′) which is not Pareto optimal.
3) There may exist a Pareto optimal RFE (p,q,z,b), but not any non–Pareto optimal RFE (p′,q′,z′,b′).
4) There may exist a Pareto optimal RFE (p,q,z,b) and also non–Pareto optimal RFE (p′,q′,z′,b′).

Next, note that there is a special case that the existence of a RAE implies always the existence of a RFE. This is the case such that $\{F_h\}$ allows the complete analogy to Arrow–Debreu economy.

PROPOSITION 6: Suppose that (A9) holds and there is at least one consumer, say $h = 1$, with $\#(F_1) = M$. Then, there exists always a RFE (p^*,q^*,z^*,b^*).

PROOF: Since financial opportunities in this case are completely characterized by Arrow securities by proposition 4, we can discuss the existence of a RFE via RAE. Let (p,π,z,a) be a RAE. Then, $a_1 = -\sum_{h\neq 1} a_h$ by definition. Next, there is at least one solution b_h^* to $a_h = [R]b_h$ for $h = 2,..,m$.

So, $\sum_{h\neq 1} a_h = \sum_{h\neq 1} [R]b_h^* = [R](\sum_{h\neq 1} b_h^*)$. Now, choose b_1^* such that $b_1^* = -\sum_{h\neq 1} b_h^*$ so that $\sum_{h\in H} b_h^* = 0$. Then, $[R]b_1^* = [R](\sum_{h\neq 1} b_h^*) = -[R](\sum_{h\neq 1} b_h^*) = -\sum_{h\neq 1} a_h = a_1$. Also, set q^* to $q^{*T} = \pi^T[R]$. Now, it is easy to see that (p^*,q^*,z^*,b^*) is a RFE by the definition 2 with $p^* = p$, $z^* = z$. Q.E.D.

Remark 8.

If $N \leq \#(F_h)$ for $h \in H$ and there is at least one consumer, say, $h = 1$, with $\#(F_1) = M$, then there exists always a Pareto optimal RFE (p,q,z,b).

Remark 9.

There is a one to one correspondence between RAE (p, π, z, a) and RFE (p,q,z,b) in this case. Since RAE (p, π, z, a) is parametrized by $\pi \in N(\pi) = \{\pi \in R^N_{++} : \sum_s \pi^s = 1\}$ here as was discussed in Lee (1987), we can study the structure of RFE allocations relative to the restricted participations taken by $h = 2,..,m$.

That is, there can be real indeterminacy of RFE allocations according to the property of the participation pattern $\{F_h\}$.

IV. CONCLUDING REMARKS

The problem about the existence of a RFE was examined in a general framework. Too many assets in financial markets will induce consumers to participate only in subsets of financial markets and this may prevent us from applying the argument about the existence of a competitive equilibrium in Arrow–Debreu economy. So, the possible extension is to characterize completely the conditions on market participations which ensures the existence of a competitive equilibrium. Also, it will be interesting to analyze what happens when assets with endogeneous returns, for instance, stocks are traded as only instruments in financial markets with restricted participations.

REFERENCES

ARROW, K. (1963–4), "The Role of Securities in the Optimal Allocation of Risk Bearing," *Review of Economic Studies*, 31, pp. 91–96.

CASS, D. (1984), "Competitive Equilibrium with Incomplete Financial Markets," Caress Working Paper, #84–09.

DEBREU, G. (1959), *Theory of Value*, New Haven: Yale University Press.

GALE, D. (1960), *The Theory of Linear Economic Models*, New York: McGraw–Hill.

GEANAKOPLOS, J. and A. MAS-COLELL (1985), "Real Indeterminacy with Financial Assets," preliminary paper, Yale University.

LEE, Y. W. (1987), "The Role of Contingent Securities Prices and Indeterminancy of Equilibria with Restrictions on Portfolio Choices," mimeo.

LEVY, H. (1978), "Equilibrium in an Imperfect Market," *American Economic Review*, 60, pp. 643–658.

SICONOLFI, P. (1986), "Equilibrium with Restricted Participation in Incomplete Financial Markets," mimeo.

WERNER, J. (1985), "Equilibrium in Economies with Incomplete financial Markets," *Journal of Economic Theory*, 36, pp. 110–119.